Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. Only write on one side of each page.

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."
– George Bernard Shaw

Problems

1. Do both of the following:
   (a) Prove that $O$ is not a normal subgroup of $M$.
   (b) Let $SM$ denote the subset of orientation-preserving motions of the plane. Prove $SM$ is a normal subgroup of $M$ and determine its index in $M$.

2. For those of you who know a bit of complex variables.
   (a) Write the formulas for the motions $t_a$, $\rho_\theta$ and $r$ in terms of the complex variables $z = x + iy$.
   (b) Show every motion has the form $m(z) = \alpha z + \beta$ or $m(z) = \alpha \bar{z} + \beta$, where $\alpha, \beta$ are complex numbers with $|\alpha| = 1$.
   (c) Find an isomorphism from the group $SM$ to the subgroup of $GL(2, \mathbb{C})$ of matrices of the form
   $$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
   with $|a| = 1$.

3. With each of the patterns shown on the sheet of figures labelled “Problem 8.3”, find a pattern with the same type of symmetry as those on the accompanying handout (the page numbered 173).

4. Given the subgroup $H = \{1, x^5\}$ of the dihedral group $D_{10}$.
   (a) Explicitly compute the cosets of $H$ in $D_{10}$.
   (b) Prove that $D_{10}/H$ is isomorphic to $D_5$.
   (c) Is $D_{10}$ isomorphic to $D_5 \times H$?

5. List all symmetries of the following figures (found on the last page of the extra-reading handout on Linear Algebra: Orthogonal Matrices and Translations).
   (a) Figure 1.4
   (b) Figure 1.5
   (c) Figure 1.6
   (d) Figure 1.7

6. Prove every finite subgroup of $M$ is a conjugate subgroup of one of the standard subgroups listed in the corollary to the Classification of Finite Symmetry Groups Theorem stated below.
Corollary 1 Let $G$ be a finite subgroup of the group of motions $M$. If coordinates are introduced suitably, then $G$ becomes one of the groups $C_n$ or $D_n$, where $C_n$ is generated by $\rho_\theta$, $\theta = 2\pi/n$ and $D_n$ is generated by $\rho_\theta$ and $r$.

7. Find all proper normal subgroups $N$ and identify the corresponding quotient groups $D_k/N$ of the groups $D_{13}$ and $D_{15}$.

8. Let $G$ be a subgroup of $M$ that contains rotations about two different points. Prove algebraically that $G$ contains a translation.

9. Prove the group of symmetries of the frieze pattern

\[
\cdots E E E E E E E E \cdots
\]

is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

10. Let $G$ be the group of symmetries of the frieze pattern

\[
\cdots \circ \circ \circ \circ \circ \circ \circ \cdots
\]

(a) Determine the point group $\bar{G}$ of $G$.
(b) For each element $\bar{g}$ of $\bar{G}$, and each element $g$ of $G$ which represents $\bar{g}$, describe the action of $g$ geometrically.
(c) Let $H$ be the subgroup of translations in $G$. Determine $[G : H]$.

11. Let $G$ be a discrete group in which every element is orientation-preserving. Prove the point group $\bar{G}$ is a cyclic group of rotations and there is a point $p$ in the plane such that the set of group elements which fix $p$ is isomorphic to $\bar{G}$.

12. Recall that $M$ is the group of rigid motions of the two-dimensional plane. In this problem you investigate the rigid motions of a one-dimensional line.

Let $N$ denote the group of rigid motions of the line $l = \mathbb{R}^1$. Some elements of $N$ are

\[
t_a \text{ where } t_a(x) = x + a \text{ and } s \text{ where } s(x) = -x.
\]

(a) Show that $\{t_a, t_as : a \in \mathbb{R}^1\}$ are all of the elements of $N$, and describe their actions on $l$ geometrically. [Note that $|N|$ is infinite since there is a distinct $t_a$ for each real number $a$.]
(b) Compute the products $t_at_b$, $st_a$, $ss$.
(c) Find all discrete subgroups of $N$ which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove your list is complete.

13. Prove

(a) If the point group of a lattice group $G$ is $\bar{G} = C_6$, then $L = L_G$ is an equilateral triangular lattice, and $G$ is the group of all rotational symmetries of $L$ about the lattice points.
(b) If the point group of a lattice group $G$ is $\bar{G} = D_6$, then $L = L_G$ is an equilateral triangular lattice, and $G$ is the group of all symmetries of $L$. 

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