Symmetric Polynomials, Pascal Matrices, and Stirling Matrices

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Several articles [3, 4, 6, 20, 21, 22] have appeared in the past two decades showing how to factorize or invert certain matrices whose entries are binomial coefficients or Stirling numbers. In this paper we present a unified approach to these results by showing that they are all special cases of results on matrices whose entries are symmetric polynomials. First we show how to factorize and invert certain matrices whose entries are the coefficients of formal power series. Applying generating functions for symmetric polynomials to these results produces factorizations and inverses of matrices whose entries are symmetric polynomials. As binomial coefficients and both kinds of Stirling numbers can be represented using symmetric polynomials we then obtain the results of the other authors as special cases.

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In the first section we review symmetric polynomials and some of their properties. Section 2 contains our main results: three theorems showing how to factorize or invert certain matrices whose entries are the coefficients of formal power series. These theorems contain as special cases many of the results on Pascal and Stirling matrices in Brawer and Pirovino [3], Call and Velleman [4], Cheon and Kim [6], and Yang and Qiao [20]. In the final section we discuss generalizations and extensions of the results in the second section; these generalize results on Pascal and Stirling matrices in Cheon and Kim [6], Zhang [21], and Zhang and Liu [22].

(Two comments on notation: In keeping with Donald Knuth's plea [13] for standardization of terminology and notation for the Stirling numbers we refer to unsigned Stirling numbers of the first kind as "Stirling cycle numbers" and Stirling numbers of the second kind as "Stirling subset numbers." Correspondingly, we refer to matrices containing these numbers as "Stirling cycle matrices" and "Stirling subset matrices." Also, indexing of rows and columns in matrices is zero-based. Thus, for example, a $k \times k$ matrix has rows and columns numbered from 0 to $k-1$.)

1 Symmetric polynomials

Given a set of variables $\{z_1, z_2, \ldots, z_n\}$, the *kth elementary symmetric polynomial* $e_k(z_1, z_2, \ldots, z_n)$ on these variables is the sum of all possible products of k of these n variables, chosen without replacement. The *kth complete symmetric polynomial* $h_k(z_1, z_2, \ldots, z_n)$ on the n variables $\{z_1, z_2, \ldots, z_n\}$ is the sum of all possible products of k of these variables, chosen with replacement. For example, the first few elementary and complete symmetric polynomials on the three variables x, y, and z are given in Table 1. The 0th symmetric polynomials we define to be 1; i.e., $e_0(z_1, z_2, \ldots, z_n) = h_0(z_1, z_2, \ldots, z_n) = 1$. The symmetric polynomials on no variables are defined to be zero; i.e., $e_k() = h_k() = 0$. These definitions contradict each other when k and n are both 0; the resolution is in

k	$e_k(x,y,z)$	$h_k(x,y,z)$
0	1	1
1	$x+y+z$	$x+y+z$
2	$xy+xz+yz$	$x^2+xy+xz+y^2+yz+z^2$
3	xyz	$x^3+x^2y+x^2z+xy^2+xyz+xz^2+y^3+y^2z+yz^2+z^3$

Table 1: Symmetric polynomials on three variables

favor of $e_0()=h_0()=1$.

Symmetric polynomials and symmetric functions have many applications. Stanley's two-volume text on combinatorics [16, 17], for example, includes an entire chapter on symmetric functions. The utility of symmetric polynomials for our purposes is that they can be used to express binomial coefficients and both kinds of Stirling numbers. From the definition of e_k it is clear that on n variables, $e_k(1,1,\ldots,1)=\binom{n}{k}$. It is also the case that on n variables, $h_k(1,1,\ldots,1)=\binom{n+k-1}{n-1}$ [16, p. 15], $e_k(1,2,\ldots,n)=\left[{n+1 \atop n-k+1}\right]$, and $h_k(1,2,\ldots,n)=\left\{{n+k \atop n}\right\}$, where $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ are Stirling cycle numbers and Stirling subset numbers, respectively [7, pp. 207-208, 213-214]. Other authors, such as Konvalina [14], Neuman [15], and Verde-Star [18], have also used symmetric functions to study binomial coefficients and Stirling numbers in a unified way.

We need the generating functions for both kinds of symmetric polynomials. These are

$$\sum_{k=0}^{\infty} e_k(z_1,z_2,\ldots,z_n)t^k=\prod_{i=1}^{n}(1+z_it)$$

[5, p. 192] and

$$\sum_{k=0}^{\infty} h_k(z_1,z_2,\ldots,z_n)t^k=\prod_{i=1}^{n}\frac{1}{1-z_it},$$

[5, p. 432]. These expressions imply that $\sum_{k=0}^{\infty} e_k(z_1,z_2,\ldots,z_n)t^k$ is a finite sum; in fact, it is easy to see from the definition that $e_k(z_1,z_2,\ldots,z_n)=0$ for $k>n$.

2 Toeplitz matrices and formal power series

Let $a(t) = \sum_{k=0}^{\infty} \alpha_k t^k$ be a formal power series. Let M_a be the infinite Toeplitz matrix whose (i,j) entry is α_{i-j} if $i \geq j$ and 0 otherwise. Thus column k of M_a is comprised of the Taylor coefficients of the series $t^k a(t)$, and row k of M_a is $\begin{bmatrix} \alpha_k & \alpha_{k-1} & \cdots & \alpha_0 & 0 & 0 & \cdots \end{bmatrix}$, the coefficients, in reverse order, of the Taylor series of $a(t)$ up to order k. It is known that $M_b M_a = M_{ba}$ and that $M_a^{-1} = M_{1/a}$ [12, pp. 14–16].

Let U be an infinite matrix. For $k \geq 1$, let $U^{(k)}$ denote the matrix obtained from U by substituting rows $0, 1, \ldots, k-1$ of I, the infinite identity matrix, for those of U. Let $U^{[k]}$ denote the matrix obtained from U be substituting columns $0, 1, \ldots, k-1$ of I for those of U. Finally, let $[U]_n$ denote the first $n \times n$ principal minor of U.

The following two theorems are our main results.

Theorem 1. *Let $a_0, a_1, a_2, \ldots$ be a sequence of formal power series of the form $a_i = \alpha_{i,0} + \alpha_{i,1} t$. Let $c_m = a_0 a_1 a_2 \cdots a_m$. Let $V = \cdots M_{a_n}^{(n)} \cdots M_{a_1}^{(1)} M_{a_0}$. Then row i of V is equal to row i of M_{c_i}.*

Proof. The infinite product V is well-defined, as entry (i,j) in V is obtained via a finite product: For $n > i$ row i of $M_{a_n}^{(n)}$ is equal to row i of I. Let $V_n = M_{a_n}^{(n)} \cdots M_{a_1}^{(1)} M_{a_0}$. We first show, by induction on n, that row i of V_n is equal to row i of $M_{c_{\min\{i,n\}}}$. Clearly the claim is true when $n = 0$. Suppose the claim holds for some $n \geq 0$. If $i < n+1$, then row i of $M_{a_{n+1}}^{(n+1)}$ is equal to row i of I. Thus row i of $\prod_{k=0}^{n+1} M_{a_{n+1-k}}^{(n+1-k)}$ is equal to row i of $\prod_{k=0}^{n} M_{a_{n-k}}^{(n-k)}$, which is equal to row i of M_{c_i}. If $i \geq n+1$, then row i of $M_{a_{n+1}}^{(n+1)}$ is equal to row i of $M_{a_{n+1}}$. Thus the only nonzero entries in row i of $M_{a_{n+1}}^{(n+1)}$ are in column n or greater. But rows n and greater in $\prod_{k=0}^{n} M_{a_{n-k}}^{(n-k)}$ are the same as rows n and greater in M_{c_n}. Thus row i of $\prod_{k=0}^{n+1} M_{a_{n+1-k}}^{(n+1-k)}$ is equal to row i of $M_{a_{n+1}} M_{c_n} = M_{a_{n+1} a_0 a_1 \cdots a_n} = M_{c_{n+1}}$. Letting n approach infinity produces the result. □

Theorem 2. *Let $a_0, a_1, a_2, \ldots$ be a sequence of formal power series. Let $c_m = a_0 a_1 a_2 \cdots a_m$. Let $W = M_{a_0} M_{a_1}^{[1]} \cdots M_{a_n}^{[n]} \cdots$. Then column j of W is equal to column j of M_{c_j}.*

Proof. The infinite product W is well-defined for reasons similar to that of V in Theorem 1. Let $W_n = M_{a_0} M_{a_1}^{[1]} \cdots M_{a_n}^{[n]}$. We show, by induction on n, that column j of W_n is equal to column j of $M_{c_{\min\{j,n\}}}$. Certainly the claim is true when $n = 0$. Suppose the claim is true for some $n \geq 0$. If $j < n+1$ then column j of $M_{a_{n+1}}^{[n+1]}$ is equal to column j of I. Thus column j of $\prod_{k=0}^{n+1} M_{a_k}^{[k]}$ is equal to column j of $\prod_{k=0}^{n} M_{a_k}^{[k]}$, which is the same as column j of M_{c_j}. If $j \geq n+1$ then column j of $M_{a_{n+1}}^{[n+1]}$ is equal to column j of $M_{a_{n+1}}$. Because $M_{a_{n+1}}^{[n+1]}$ is lower-triangular the elements in columns $0, 1, \ldots, j-1$ of $\prod_{k=0}^{n} M_{a_k}^{[k]}$ are irrelevant for calculating the product of row i in $\prod_{k=0}^{n} M_{a_k}^{[k]}$ and column j of $M_{a_{n+1}}^{[n+1]}$. Since $j \geq n+1$ and columns n and greater in $\prod_{k=0}^{n} M_{a_k}^{[k]}$ are equal to those of M_{c_n}, we have that column j of $\prod_{k=0}^{n+1} M_{a_k}^{[k]}$ is equal to column j of $M_{c_n} M_{a_{n+1}} = M_{c_{n+1}}$. Letting n approach infinity produces the result. □

Many of the factorization and inverse results in Brawer and Pirovino [3], Call and Velleman [4], Cheon and Kim [6], and Yang and Qiao [20] on Pascal and Stirling matrices are special cases of Theorems 1 or 2 or one of their corollaries applied to the first principal minor of V or W, as we now show.

Let $z_1, z_2, \ldots$ be a sequence of variables, and define

$$(F(z_1, z_2, \ldots))_{ij} = \begin{cases} e_{i-j}(z_1, z_2, \ldots, z_i), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 1. $F(z_1, z_2, \ldots) = \cdots M_{1+z_n t}^{(n)} \cdots M_{1+z_1 t}^{(1)}$.

Proof. Using the notation of Theorem 1, let $a_0 = 1$, $a_i = 1 + z_i t$ for $i \geq 1$. The coefficient of t^k in c_n is $e_k(z_1, z_2, \ldots, z_n)$. Theorem 1 then implies that, for $i \geq j$, entry (i, j) in $\cdots M_{1+z_n t}^{(n)} \cdots M_{1+z_1 t}^{(1)} M_1$ is $e_{i-j}(z_1, z_2, \ldots, z_i)$. Finally, $M_1 = I$. □

Entry (i,j) in $F(1,1,\ldots)$ is $\binom{i}{i-j}$. Thus $[F(1,1,\ldots)]_n$ is the $n\times n$ Pascal matrix. In fact, setting $z_i=1$, applying Corollary 1 and extracting the first $n\times n$ upper left submatrix of each factor yields the factorization of the Pascal matrix given explicitly in Yang and Qiao [20] and implicitly (via the inverse of the Pascal matrix) in Brawer and Pirovino [3] and Zhang [21]. Similarly, $[F(1,2,3,\ldots)]_n$ is the $n\times n$ Stirling cycle matrix, and Corollary 1 yields a corresponding factorization. (Incidentally, this factorization does not appear in work by any of the aforementioned authors.)

Define

$$(G(z_1,z_2,\ldots))_{ij}=\begin{cases}h_{i-j}(z_1,z_2,\ldots,z_{j+1}), & \text{if } i\geq j;\\ 0, & \text{otherwise.}\end{cases}$$

Corollary 2. $G(z_1,z_2,\ldots)=M_{1/(1-z_1t)}M^{[1]}_{1/(1-z_2t)}\cdots M^{[n-1]}_{1/(1-z_nt)}\cdots.$

Proof. Let $a_i=1/(1-z_{i+1}t)$ for $i\geq 0$. Then the coefficient of t^k in the power series representation of c_n is $h_k(z_1,z_2,\ldots,z_{n+1})$. Theorem 2 then implies that, for $i\geq j$, entry (i,j) in $M_{1/(1-z_1t)}M^{[1]}_{1/(1-z_2t)}\cdots M^{[n-1]}_{1/(1-z_nt)}\cdots$ is $h_{i-j}(z_1,z_2,\ldots,z_{j+1})$. □

Entry (i,j) in $G(1,1,\ldots)$ is $\binom{j+1+i-j-1}{j+1-1}=\binom{i}{j}$. Thus $[G(1,1,\ldots)]_n$ is also the $n\times n$ Pascal matrix. With $z_i=1$ and using the first $n\times n$ upper left submatrices Corollary 2 yields the most common factorization of the Pascal matrix; it is given explicitly in Brawer and Pirovino [3], Yang and Qiao [20], and Zhang [21]. In addition, $[G(1,2,3,\ldots)]_n$ is the $n\times n$ Stirling subset matrix, and Corollary 2 gives a (apparently new) factorization of this matrix.

The results on inverses of Pascal and Stirling matrices in the work of these other authors are special cases of Theorems 1 and 2 and the following result.

Theorem 3. *Let* $a(t)=1+\alpha_1t$. *Then* $M^{(n)}_aM^{[n-1]}_{1/a}=I$.

Proof. For rows and columns $n-2$ and smaller $M^{(n)}_a$ and $M^{[n-1]}_{1/a}$ are the same as I. For rows and columns n and greater $M^{(n)}_a$ is the same as the lower bidiagonal matrix M_a, and for rows and columns $n-1$ and larger $M_{1/a}^{[n-1]}$ is the same as the lower triangular matrix $M_{1/a}$. Thus row i, $0 \le i < n-1$, of $M_a^{(n)} M_{1/a}^{[n-1]}$ is the same as that of I, and row i, $i \ge n$, is the same as that of $M_a M_{1/a} = M_1 = I$. All of the entries in row $n-1$ of $M_a^{(n)} M_{1/a}^{[n-1]}$ are zero, except for entry $(n-1, n-1)$, which is the constant coefficient of $1/a$. However, since the constant coefficient of a is 1, the constant coefficient of $1/a$ must also be 1. □

Theorem 3 and Corollaries 1 and 2 immediately imply the following.

Corollary 3. *$F(z_1, z_2, \ldots)G(-z_1, -z_2, \ldots) = I$.*

With $z_i = 1$ for all i in Corollary 3 we have the fact that the Pascal matrix is (up to sign) its own inverse. Similarly, setting $z_i = i$ in Corollary 3 yields the result that the Stirling cycle and subset matrices are (again, up to sign) inverses of each other.

As mentioned, the factorizations of the Stirling matrices yielded directly by Corollaries 1 and 2 are different from those obtained by the authors considered here. Instead, the factorizations given by Cheon and Kim [6] and Yang and Qiao [20] are in terms of Pascal matrices. However, these factorizations also follow – in a different way – from Corollaries 1 and 2 and Theorem 3, which we now show.

The infinite generalized Pascal matrix $P(z)$ (in the manner of Call and Velleman [4] or Zhang [21]) is given by

$$(P(z))_{ij} = \begin{cases} z^{i-j}\binom{i}{j}, & \text{if } i \ge j; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $P(z)$ is equal to both $F(z, z, z, \ldots)$ and $G(z, z, z, \ldots)$. Given a matrix A, define $\bar{A}_n$ to be the matrix $\begin{bmatrix} I_n & 0 \\ 0 & A \end{bmatrix}$. Then we have the following:

Corollary 4. *$\bar{P}_1(-z)P(z) = M_{1+zt}^{(1)}$, and $P(z)\bar{P}_1(-z) = M_{1/(1-zt)}$.*

Proof. Corollary 2 says that $\bar{P}_1(-z) = \bar{G}_1(-z, -z, -z, \ldots) = M^{[1]}_{1/(1+zt)} \cdots M^{[n-1]}_{1/(1+zt)} \cdots$, and Corollary 1 yields $P(z) = F(z, z, z, \ldots) = \cdots M^{(n)}_{1+zt} \cdots M^{(1)}_{1+zt}$. However, row i of $\bar{P}_1(-z)$ is determined solely by the product $M^{[1]}_{1/(1+zt)} \cdots M^{[i-1]}_{1/(1+zt)}$. (Matrix $M^{[i]}_{1/(1+zt)}$ is not included because entry (i, i) is a 1, and thus it does not affect the product.) Similarly, entries 0 through i in column j of $P(z)$ are determined solely by the product $M^{(i)}_{1+zt} \cdots M^{(1)}_{1+zt}$. Thus entry (i, j) in $\bar{P}_1(-z)P(z)$ is entry (i, j) in the product $M^{[1]}_{1/(1+zt)} \cdots M^{[i-1]}_{1/(1+zt)} M^{(i)}_{1+zt} \cdots M^{(1)}_{1+zt}$. By Theorem 3, though, this is $M^{(1)}_{1+zt}$. Similarly, the factorizations of $P(z) = G(z, z, z, \ldots)$ and $\bar{P}_1(-z) = \bar{F}_1(-z, -z, -z, \ldots)$ yield $P(z)\bar{P}_1(-z) = M_{1/(1-zt)}$. □

Corollary 5. *$F(z_1, z_2, \ldots) = \cdots \bar{P}_n(z_{n+1} - z_n) \cdots \bar{P}_1(z_2 - z_1)P(z_1)$ and $G(z_1, z_2, \ldots) = P(z_1)\bar{P}_1(z_2 - z_1) \cdots \bar{P}_n(z_{n+1} - z_n) \cdots$.*

Proof. We use the fact that $P(z_1)P(z_2) = P(z_1 + z_2)$, given in Call and Velleman [4]. By Corollaries 1 and 4 we have

$$\begin{aligned} F(z_1, z_2, \ldots) &= \cdots M^{(n)}_{1+z_n t} M^{(n-1)}_{1+z_{n-1} t} \cdots M^{(1)}_{1+z_1 t} \\ &= \cdots \bar{P}_n(-z_n)\bar{P}_{n-1}(z_n)\bar{P}_{n-1}(-z_{n-1})\bar{P}_{n-2}(z_{n-2}) \cdots \bar{P}_1(-z_1)P(z_1) \\ &= \cdots \bar{P}_n(z_{n+1} - z_n)\bar{P}_{n-1}(z_n - z_{n-1}) \cdots \bar{P}_1(z_2 - z_1)P(z_1). \end{aligned}$$

A similar argument proves the factorization $G(z_1, z_2, \ldots) = P(z_1)\bar{P}_1(z_2 - z_1) \cdots \bar{P}_n(z_{n+1} - z_n) \cdots$. □

Setting $z_i = i$ for each i in Corollary 5 yields the factorizations $F(1, 2, 3, \ldots) = \cdots \bar{P}_n(1) \cdots \bar{P}_1(1)P(1)$ and $G(1, 2, 3, \ldots) = P(1)\bar{P}_1(1) \cdots \bar{P}_n(1) \cdots$ of the Stirling cycle and subset matrices, respectively. These are given (in their finite matrix versions) either explicitly or implicitly in Cheon and Kim [6] and Yang and Qiao [20].

3 Generalizations and extensions

In the same manner we defined generalized Pascal matrices we can define generalized Stirling matrices (see, for example, Cheon and Kim [6]) and even generalized F and G matrices:

$$(F(x; z_1, z_2, \ldots))_{ij} = \begin{cases} x^{i-j} e_{i-j}(z_1, z_2, \ldots, z_i), & \text{if } i \geq j; \\ 0, & \text{otherwise;} \end{cases}$$

$$(G(x; z_1, z_2, \ldots))_{ij} = \begin{cases} x^{i-j} h_{i-j}(z_1, z_2, \ldots, z_{j+1}), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

A generalized matrix such as one of these can actually be expressed quite simply in terms of the original matrix. Define D_x to be the diagonal matrix with the sequence $1, x, x^2, x^3, \ldots$ on the diagonal. Then we have the following.

Lemma 1. *Let A and $A(x)$ be matrices such that entry (i, j) in $A(x)$ is given by $x^{i-j} a_{ij}$, where a_{ij} is entry (i, j) in A. Then $A(x) = D_x A D_x^{-1}$.*

Proof. Left multiplication of A by D_x has the effect of multiplying row i of A by x^i. Right multiplication of A by D_x^{-1} has the effect of multiplying column j by $1/x^j$. Thus entry (i, j) in $D_x A D_x^{-1}$ is $x^{i-j} a_{ij}$. □

Let $M_a(x)$ be the generalized M_a matrix. By applying Lemma 1 with results from the preceding section we obtain the following.

Corollary 6.

$$\begin{aligned} F(x; z_1, z_2, \ldots) &= \cdots M^{(n)}_{1+z_n t}(x) \cdots M^{(1)}_{1+z_1 t}(x) \\ &= \cdots \bar{P}_n((z_{n+1} - z_n)x) \cdots \bar{P}_1((z_2 - z_1)x) P(z_1 x) \\ G(x; z_1, z_2, \ldots) &= M_{1/(1-z_1 t)}(x) M^{[1]}_{1/(1-z_2 t)}(x) \cdots M^{[n-1]}_{1/(1-z_n t)}(x) \cdots \\ &= P(z_1 x) \bar{P}_1((z_2 - z_1)x) \cdots \bar{P}_n((z_{n+1} - z_n)x) \cdots \end{aligned}$$

$$F(x; z_1, z_2, \ldots) G(-x; z_1, z_2, \ldots) = I.$$

Factorizations and inverses of generalized Pascal and Stirling matrices found in Zhang [21] and Cheon and Kim [6] can be obtained directly from Corollary 6; again, set $z_i = 1$ for all i to obtain results for the generalized Pascal matrix and set $z_i = i$ for each i to obtain results for the generalized Stirling matrices.

In the manner of Zhang and Liu [22] we can define extended generalized versions of the F and G matrices as well. Let

$$(F(x, y; z_1, z_2, \ldots))_{ij} = \begin{cases} x^{i-j} y^{i+j} e_{i-j}(z_1, z_2, \ldots, z_i), & \text{if } i \geq j; \\ 0, & \text{otherwise;} \end{cases}$$

$$(G(x, y; z_1, z_2, \ldots))_{ij} = \begin{cases} x^{i-j} y^{i+j} h_{i-j}(z_1, z_2, \ldots, z_{j+1}), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

As with the generalized matrices, an extended generalized matrix can be easily expressed in terms of the original matrix.

Lemma 2. *Let A and $A(x, y)$ be matrices such that entry (i, j) in $A(x, y)$ is given by $x^{i-j} y^{i+j} a_{ij}$, where a_{ij} is entry (i, j) in A. Then $A(x, y) = D_y D_x A D_x^{-1} D_y$.*

Proof. The proof is virtually identical to that of Lemma 1. □

By applying Lemmas 1 and 2 with results from the preceding section we have the following.

Corollary 7.

$$\begin{aligned} F(x, y; z_1, z_2, \ldots) &= \cdots M^{(n)}_{1+z_n t}(xy) \cdots M^{(1)}_{1+z_1 t}(xy) D_y^2 \\ &= D_y^2 \left(\cdots M^{(n)}_{1+z_n t}(x/y) \cdots M^{(1)}_{1+z_1 t}(x/y) \right) \\ &= \cdots \bar{P}_n((z_{n+1} - z_n)xy) \cdots \bar{P}_1((z_2 - z_1)xy) P(z_1 xy) D_y^2 \\ &= D_y^2 \left(\cdots \bar{P}_n((z_{n+1} - z_n)x/y) \cdots \bar{P}_1((z_2 - z_1)x/y) P(z_1 x/y) \right) \end{aligned}$$

$$
\begin{aligned}
G(x,y;z_1,z_2,\ldots) &= \left(M_{1/(1-z_1t)}(xy)M^{[1]}_{1/(1-z_2t)}(xy)\cdots M^{[n-1]}_{1/(1-z_nt)}(xy)\cdots\right)D_y^2 \\
&= D_y^2 M_{1/(1-z_1t)}(x/y)M^{[1]}_{1/(1-z_2t)}(x/y)\cdots M^{[n-1]}_{1/(1-z_nt)}(x/y)\cdots \\
&= \left(P(z_1xy)\bar{P}_1((z_2-z_1)xy)\cdots\bar{P}_n((z_{n+1}-z_n)xy)\cdots\right)D_y^2 \\
&= D_y^2 P(z_1x/y)\bar{P}_1((z_2-z_1)x/y)\cdots\bar{P}_n((z_{n+1}-z_n)x/y)\cdots
\end{aligned}
$$

$$
F(x,y;z_1,z_2,\ldots)G(x,-1/y;z_1,z_2,\ldots) = I.
$$

Corollary 7 contains, as special cases, factorizations and inverses of extended generalized Pascal and Stirling matrices found in Zhang and Liu [22] and Cheon and Kim [6].

For more on Stirling and Pascal matrices see papers by Aceto and Trigiante [1], Bayat and Teimoori [2], Edelman and Strang [8], El-Mikkawy [9, 10, 11], or Verde-Star [19].

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