Math 211

Final Exam

Name _____

Friday, Dec. 16, Noon
200 pts.

I.   Logic and proof

1.   (15 pts.) Consider the statement "**If** all vertices of a connected graph have even degree **then** the graph has an Euler circuit"

What is the sufficient condition?

What is the necessary condition?

What is the converse of the statement?

What is the contrapositive of the statement?

As it happens, the converse of this statement is also true. Re-write the statement using the words "sufficient" and "necessary" to indicate this.3.

2. (10 pts). Construct a truth table for $((\neg q) \wedge (p \Rightarrow q)) \Rightarrow p$. Is the statement a tautology?

3. (10 pts.)

    a. If p(x) means "x is a point", l(x) "x is a line", par(x,y) "x and y are parallel' and on(x,y) "x is on y", translate into ordinary English (i.e., not simply a transliteration):
    $\forall x \forall y (p(x) \wedge l(y) \wedge \neg on(x, y) \Rightarrow \exists z (l(z) \wedge on(x, z) \wedge par(y, z)))$

    b. Suppose F(x) means "x is a bird", B(x) "x is brightly colored", and G(x) "x is ground-feeding" Translate the statement "No brightly colored bird is a ground feeder" into logical notation.

4. (5 pts.) Simplify

$$\neg \forall x (S(x) \Rightarrow T(x))$$

so that **not** ($\neg$)does not appear [except next to a predicate]

5. (10 pts.) Prove that $A \cap (B \cup C) = ((A \cap B) \cup (A \cap C))$

II.   Algorithms, functions, and asymptotic behavior

1.   (5 pts.) Give a brief definition of an **algorithm**

2.   (10 pts.)

What do we mean by a function f:A -> B?

Let f(x) = x+1 and let g(x) = 2x. What is

$(f \circ g)(x)$?

(f+g)(x)?

3.   (10 pts.)   Give both a formal and an informal definition of what it means to say that f is O(g)

4.   (10 pts.)   For each of the following algorithms, give the best "big-O" estimate of its cost in time as a function of the size of the array or graph:

linear search

bubble sort

binary search

Finding a shortest path between two vertices of a graph

Finding a Hamilton circuit in a graph

QuickSort

III. Number theory (with some incorporated material on relations)

1. (10 pts.) Consider the integers mod 5. We define

[k] = {n in Z (the integers) | n _ k mod 5}

a. Why are there only five distinct such sets ([0], [1], [2], [3], [4])

b. We can define addition and multiplication on these three sets as follows:
    [j] + [k] = [j+k]
    [j] * [k] = [j*k] (as in a lecture past)

complete the following addition and multiplication tables using only [0], [1], [2], [3], and [4]:

+	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

*	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

What is 1/[2]?

2. (10 pts.)

    a. Define a|b (evenly divides)

    b. Define a relation on the integers by a|b. Describe the properties of this relation (i.e, symmetric, anti-symmetric, transitive, etc.)

IV. Induction

    1. (10 pts.) Prove that $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$ where f_k is the kth Fibonacci number ($f_1=1$, $f_2=1$, $f_3=2$, $f_4=3$, ...). Carefully describe each step.

V.   Counting and probability

1.   (5 pts.) What is the coefficient of $x^7 y^3$ in the expansion of $(x + y)^{10}$ (please provide a numeric answer).

2.   (10 pts.)
        In how many ways can you throw a [sum of] 4 using a pair of fair dice?

        What is the probability of doing so?

3.   (10 pts.)

How many ways can you draw a five-card hand from a deck of 52 cards so that the hand contains exactly one ace?

What is the probability of doing so?

4.   (10 pts.) In how many ways can you place 10 indistinguishable marbles into three boxes?

At least one box has at least how many marbles? What principle are you using?

VI   Relations and databases (10 pts.)

    a. (10 pts.) What does it mean for a relation $R \subseteq A X A$ to be

       reflexive

       symmetric

       anti-symmetric

       transitive

       an equivalence relation

b. (15 pts.)   Consider the usual supplier-parts-project database

S(SNO, SNAME, STATUS, CITY)
P(PNO, PNAME, COLOR, WEIGHT, CITY)
J(JNO, JNAME, CITY)
SPJ(SNO, PNO, JNO, QTY)

In the relational algebra,

What does the **projection** operator do?

What does the **selection** operator do?

What does the **join** operator do?

c. In the **SQL** statement below, identify occurrences of the **projection**, **selection**, and **join** operations:

SELECT SNAME
       FROM S, P, SPJ
       WHERE P.COLOR = "RED"
             AND P.PNO = SPJ.PNO
   AND S.SNO = SPJ.SNO;

VII.   Graphs

1.   (10 pts.) Sketch the graph resulting from the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The adjacency matrix counts the number of paths of length 1 between vertices. Calculate the matrix that counts the number of paths of length 2 between vertices.

Construct an incidence matrix for the above graph.

2.   (10 pts.)   Use Dijkstra's algorithm to find a shortest path between A and Z. Label each node with the final L value of that node.

A —2— B
| \4
1   \
| 1   Z
C —4— D —1—
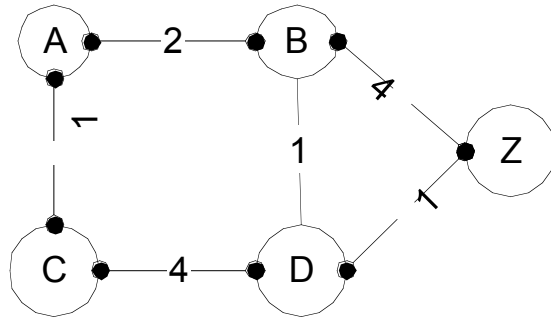
VIII. History and famous problems (10 pts.) Pick four of the following names of people and mathematical problems, and say something about them. Clearly indicate which person/problem you are describing (duplicates do not count if I have accidentally included one).

a)   René Descartes
b)   Georg Cantor
c)   The 3x+1 conjecture
d)   Leonardo of Pisa
e)   Karl Friedrich Gauss
f)   Charles Dodgson
g)   Paul Gustav Heinrich Bachman
h)   Ada Augusta, Countess of Lovelace
i)   The halting problem
j)   Marin Mersenne
k)   James Bernoulli
l)   Pierre-Simon Laplace
m)   Abu Ja'far Mohammed Ibn Musa Al-Khowarizmi
n)   Donald Knuth
o)   George Boole
p)   Goldbach's conjecture
q)   Pierre de Fermat
r)   G. Lejeune Dirichlet
s)   The twin primes conjecture
t)   Pierre-Simon Laplace
u)   Paul Erdös