Using symmetry to solve differential equations

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Outline

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1. “Magic” coordinates
2. Symmetries of a differential equation
3. Using a symmetry to find “magic” coordinates

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2. Symmetries of a differential equation
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4. Finding symmetries of a differential equation

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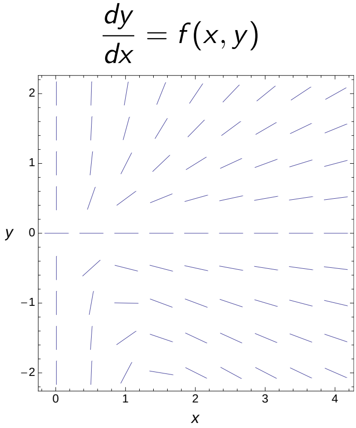

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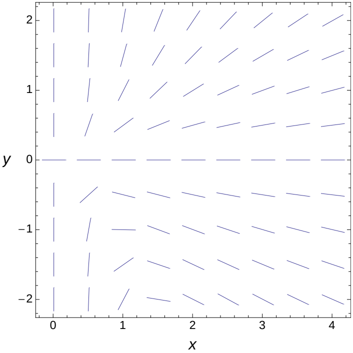

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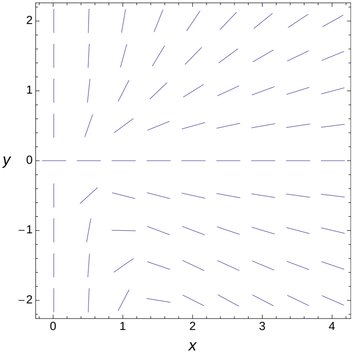

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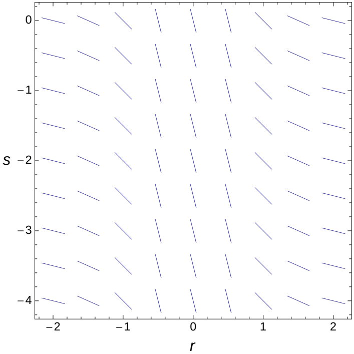

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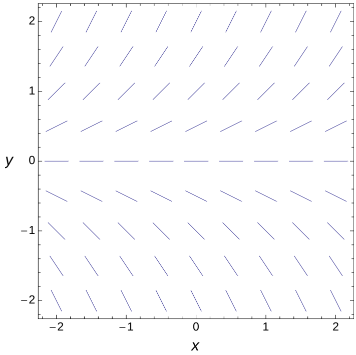

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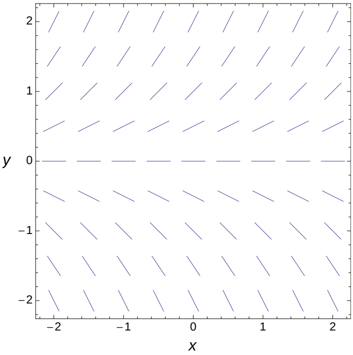

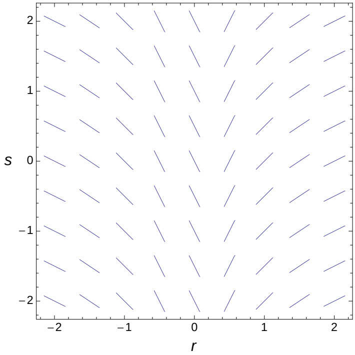

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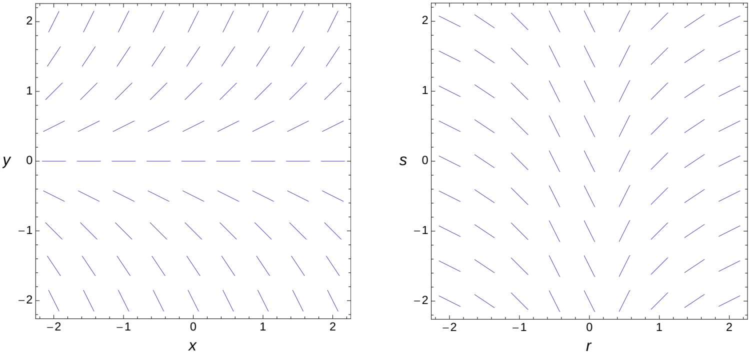

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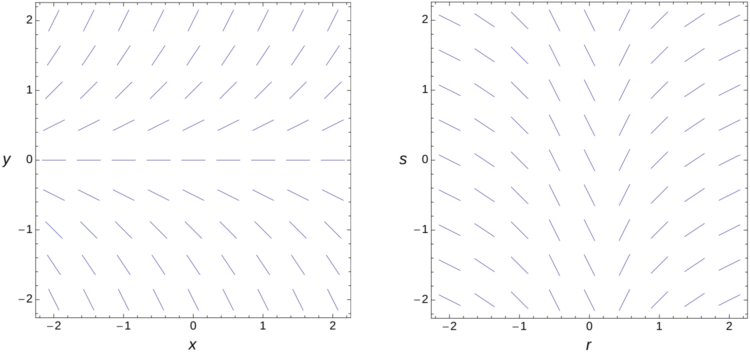

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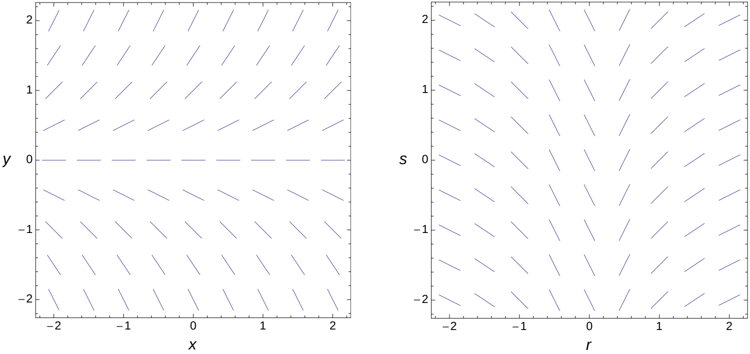

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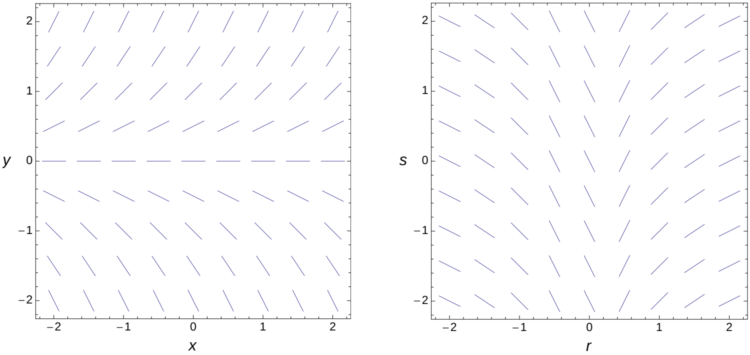

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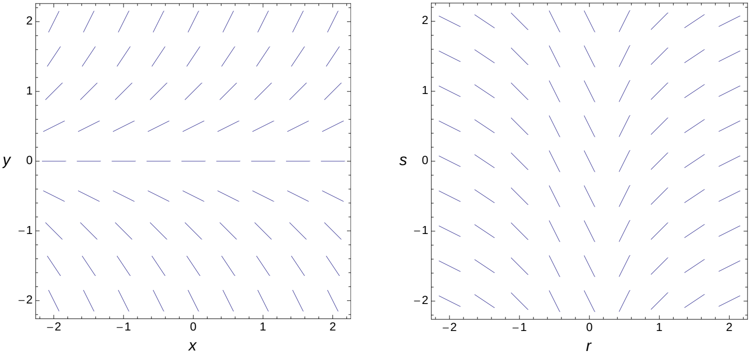

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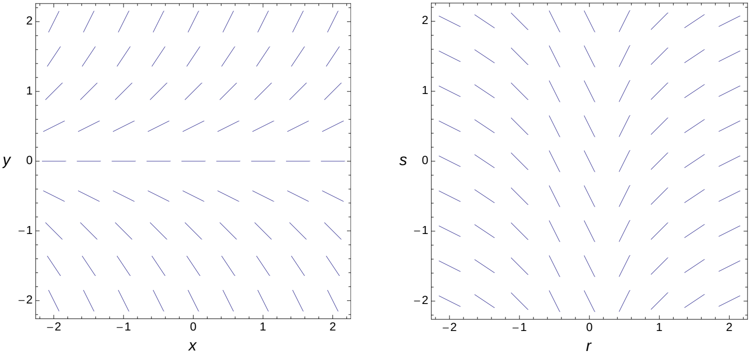

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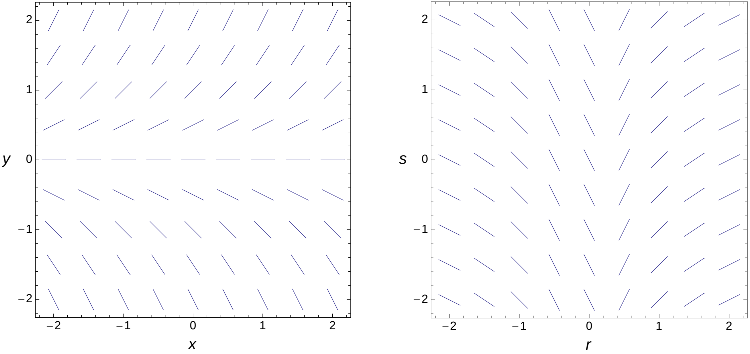

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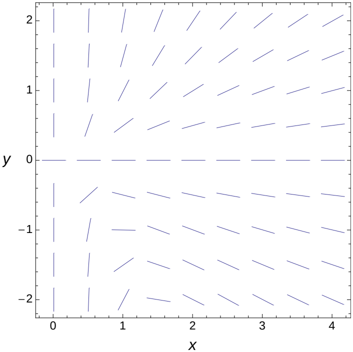

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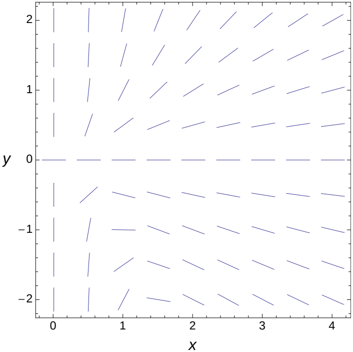

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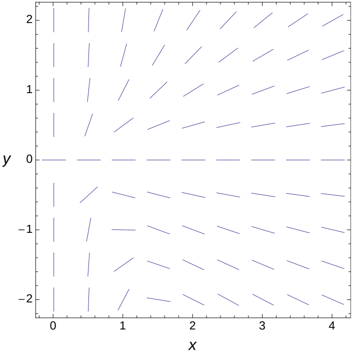

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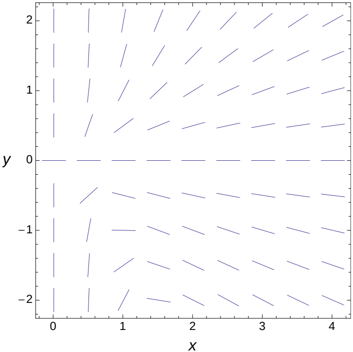

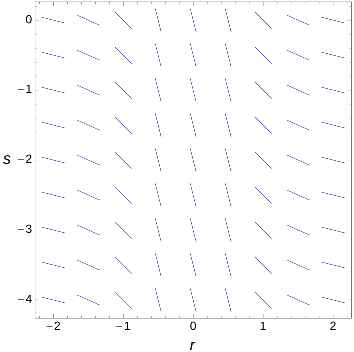

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- Answer: symmetry!

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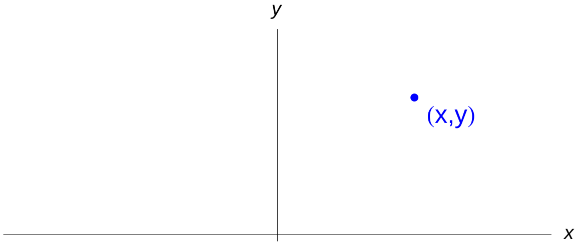

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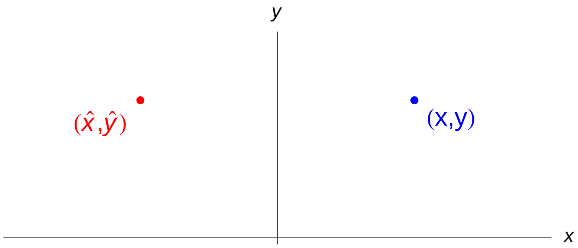

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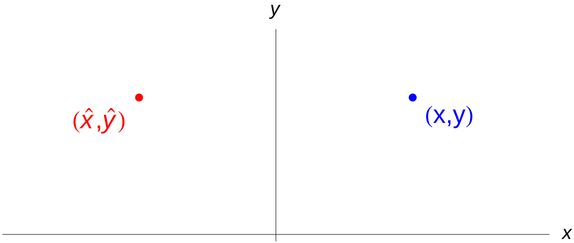

- denote this

$$(\hat{x}, \hat{y}) = T(x, y) = (-x, y)$$

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$$(\hat{x}, \hat{y}) = (x \cos\epsilon - y \sin\epsilon, x \sin\epsilon + y \cos\epsilon)$$

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- now ready to define symmetries of geometric objects

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- in general, a geometric object in the plane has a *symmetry flow* (or a *Lie symmetry*) if there is a "nice" transformation flow of the plane that maps that object to itself

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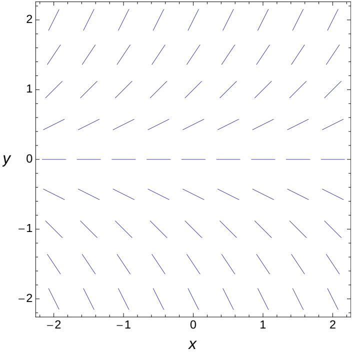

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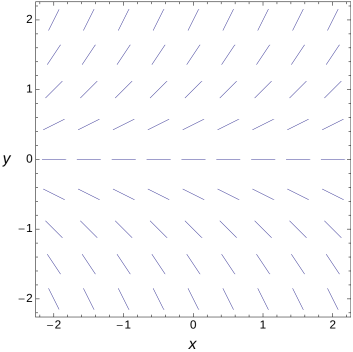

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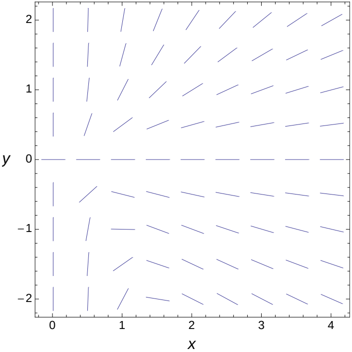

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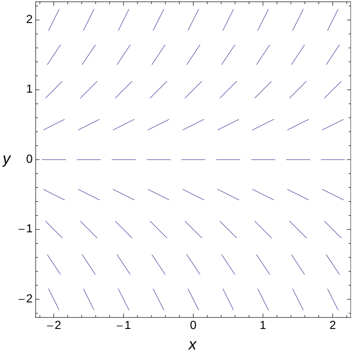

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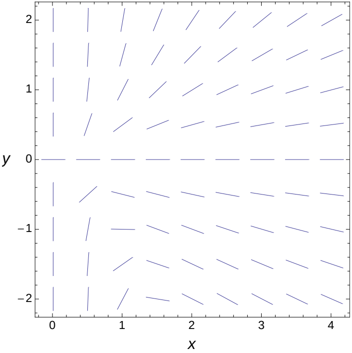

- to understand how a slope field transforms, first look at how slopes transform

Effect of a transformation on slopes

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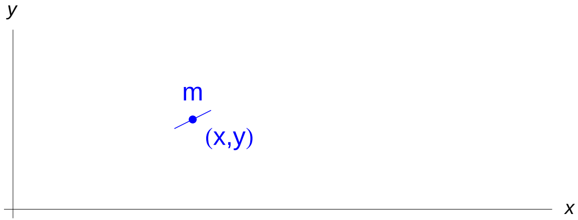

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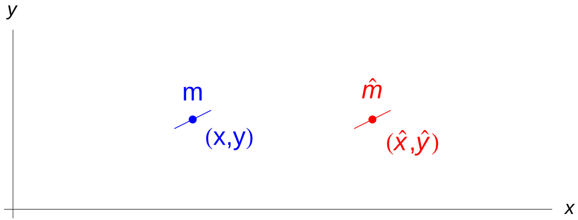

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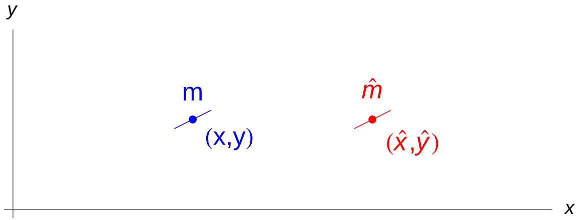

  - each transformed tangent line segment has slope $\hat{m}$ that is the same as the original slope m so

$$(\hat{x}, \hat{y}, \hat{m}) = T_\epsilon(x, y, m) = (x + \epsilon, y, m)$$

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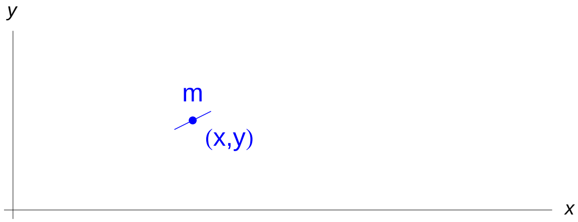

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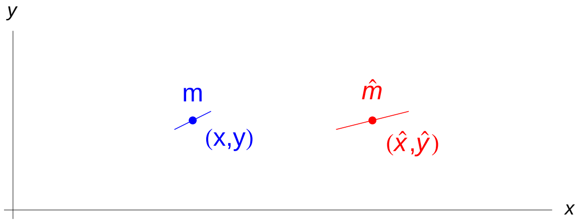

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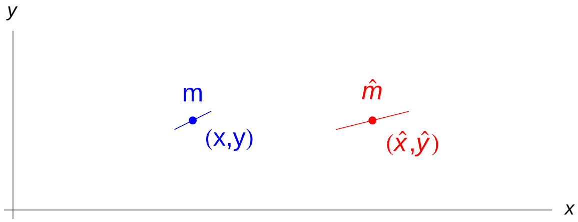

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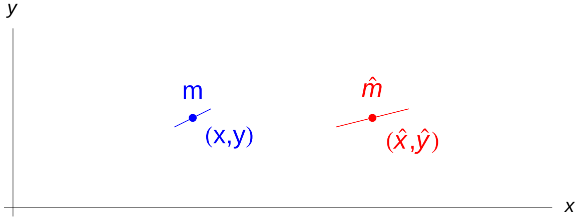

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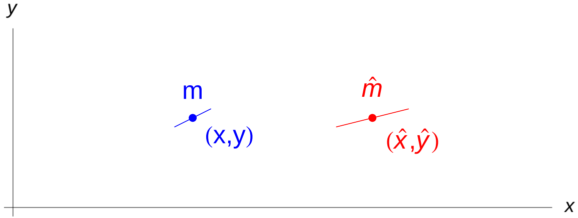

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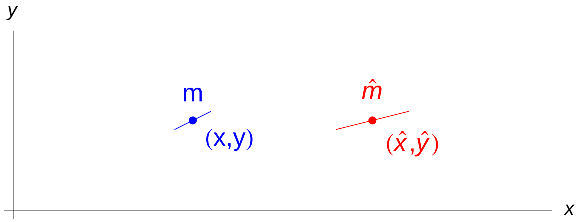

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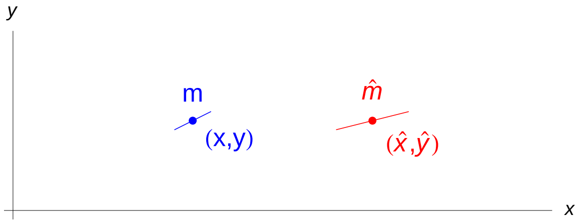

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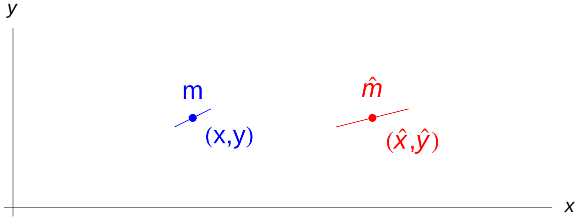

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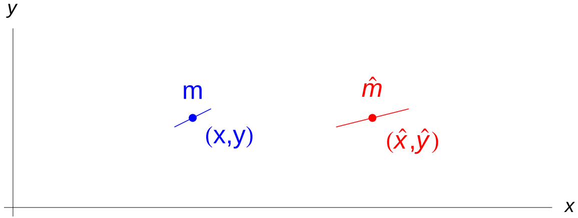

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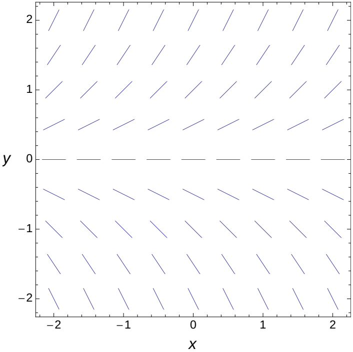

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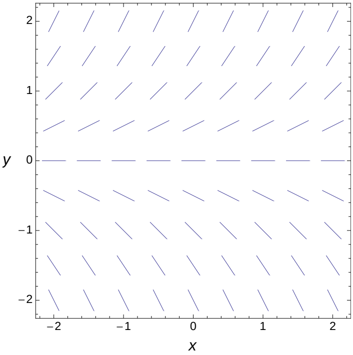

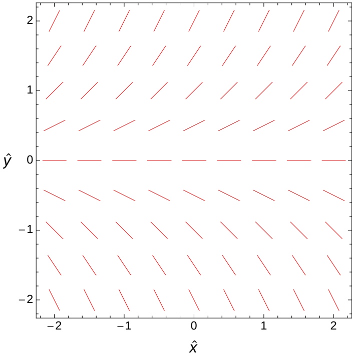

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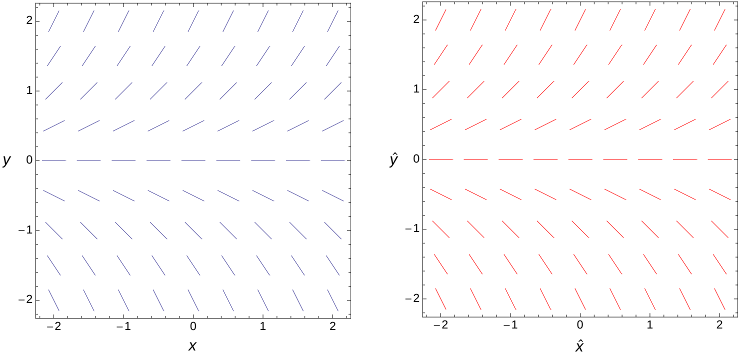

- T is a *symmetry* of the differential equation if the slope field maps to itself (so each solution is mapped to a solution)

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- before working with symmetries of a differential equation, look at a convenient way to picture a transformation flow

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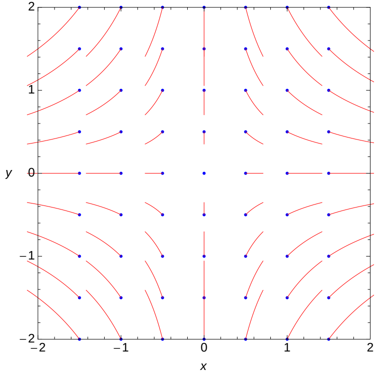

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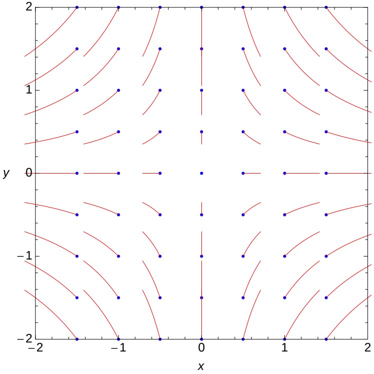

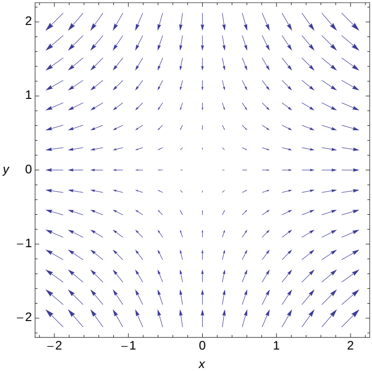

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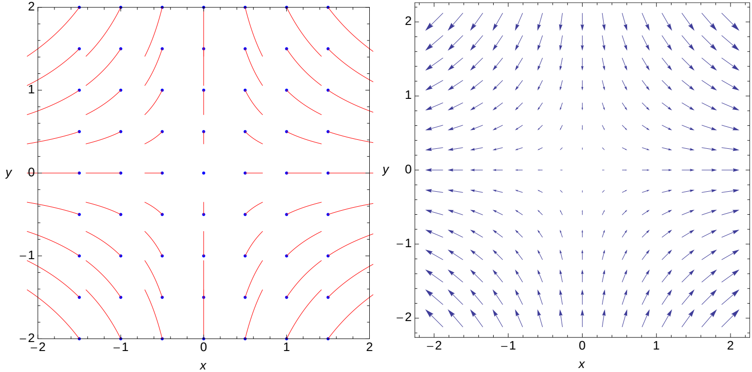

- denote the tangent vector field $\vec{X} = (\xi, \eta)$

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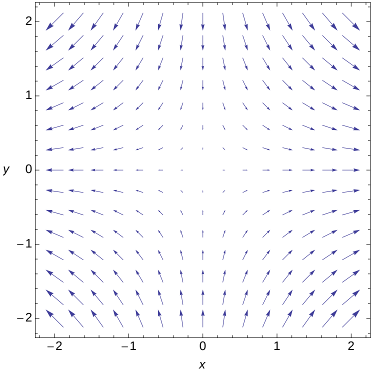

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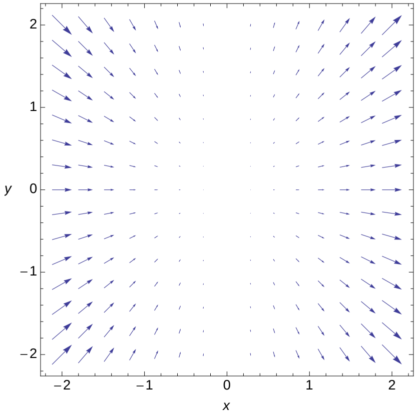

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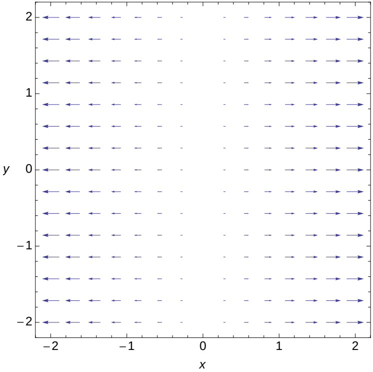

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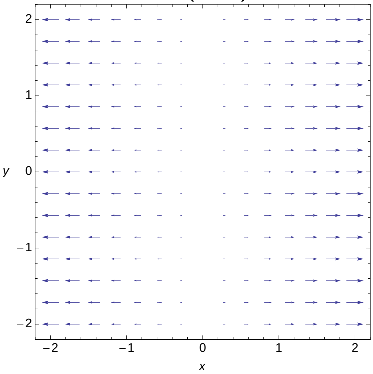
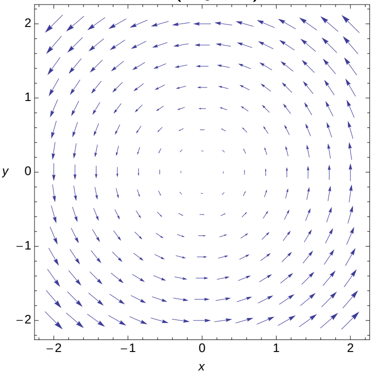


Rotation

$$(\hat{x}, \hat{y}) = (x \cos\epsilon - y \sin\epsilon, x \sin\epsilon + y \cos\epsilon)$$

$$\vec{X} = (-y, x)$$



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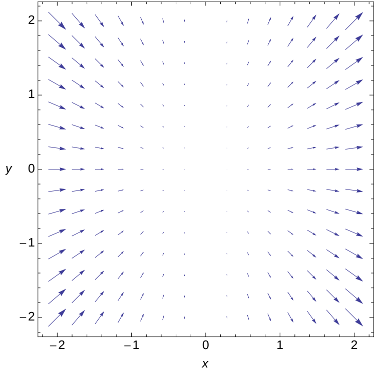

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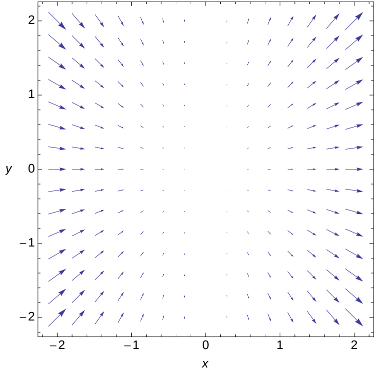

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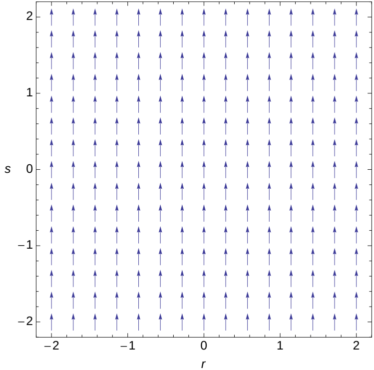

Using a symmetry to solve the differential equation

- find coordinates (r, s) in which the symmetry field is vertical and uniform

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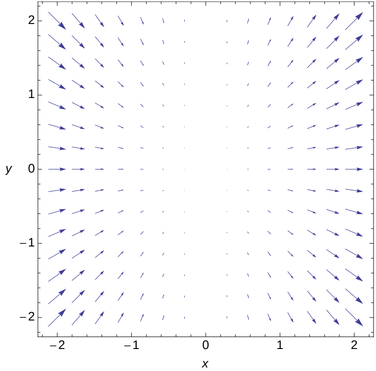

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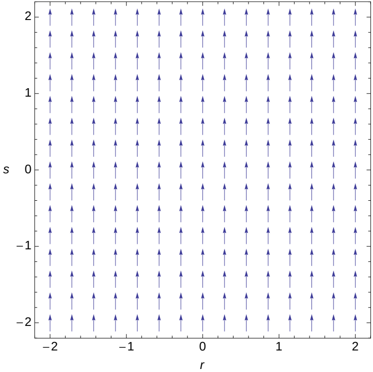

- in the new coordinates, differential equation reduces to an antiderivative problem since symmetry maps solutions to solutions by translation in the dependent variable

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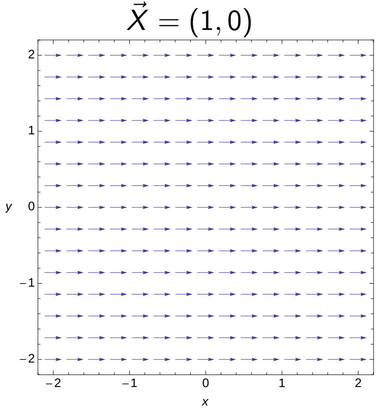

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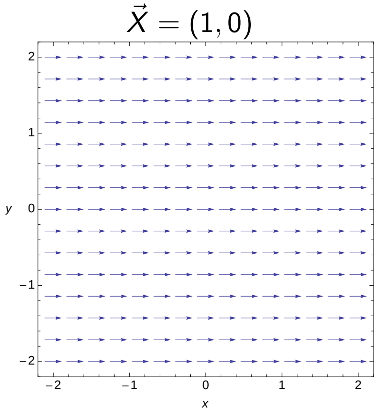

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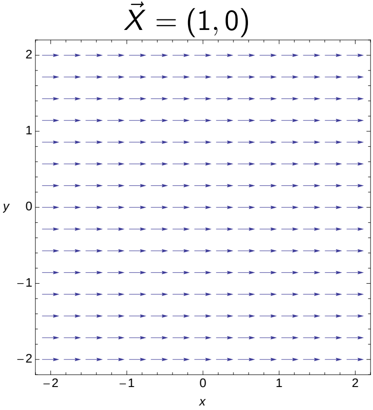

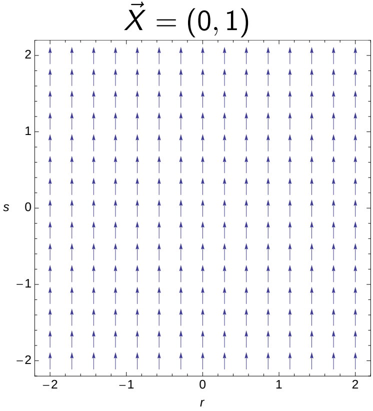

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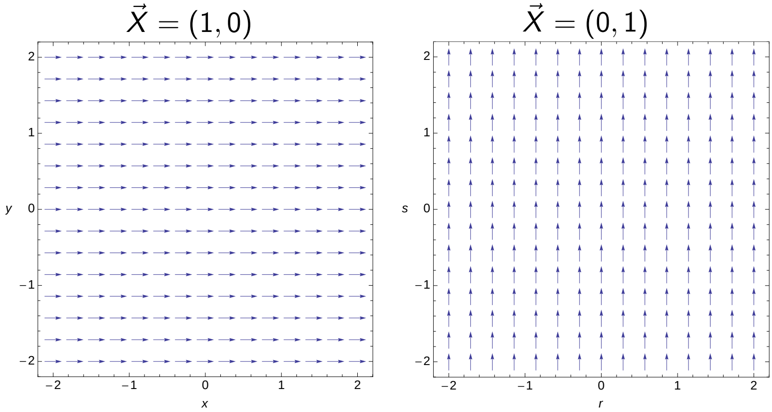

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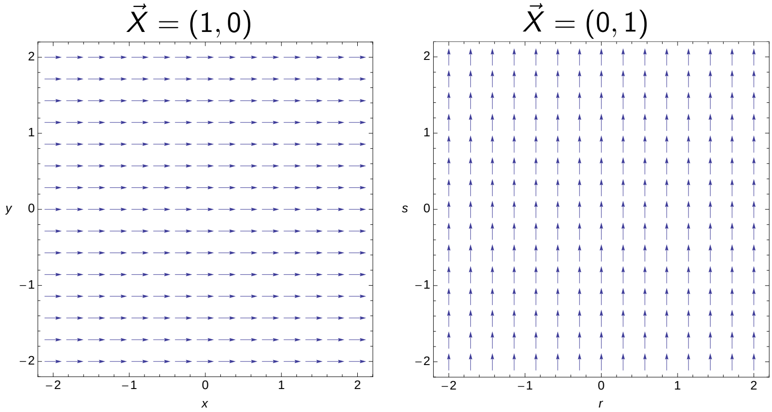

- this is a symmetry flow for $\dfrac{dy}{dx} = y$
- in these “magic coordinates”, this ODE becomes $\dfrac{ds}{dr} = \dfrac{1}{r}$

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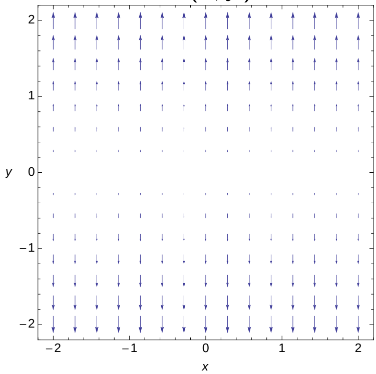

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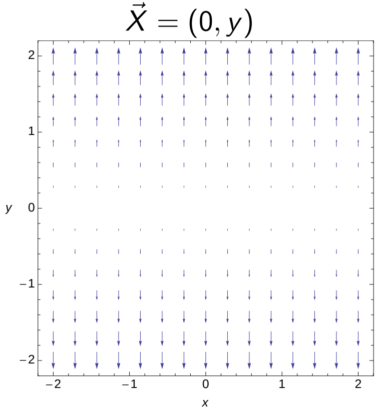

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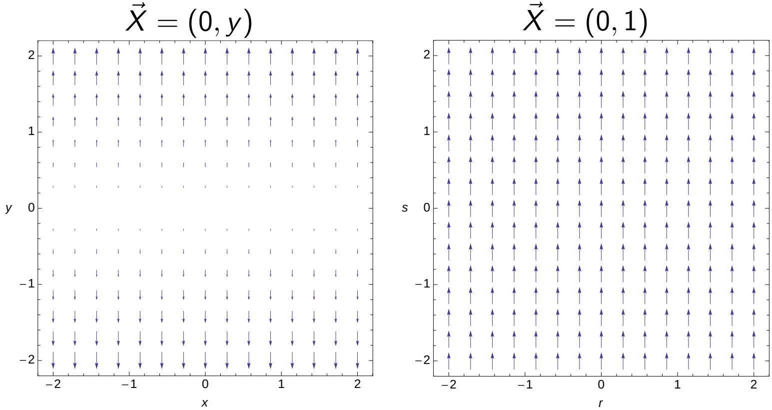

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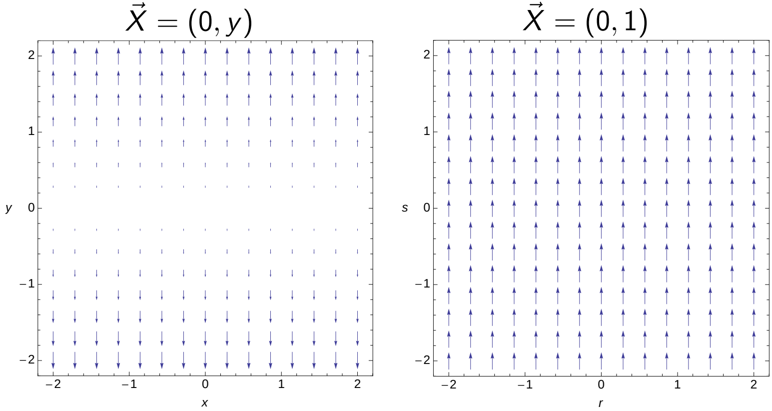

- this is also a symmetry flow for $\dfrac{dy}{dx} = y$ so can now transform the differential equation to the new coordinates

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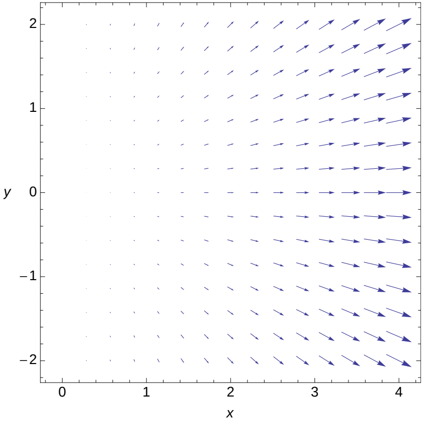

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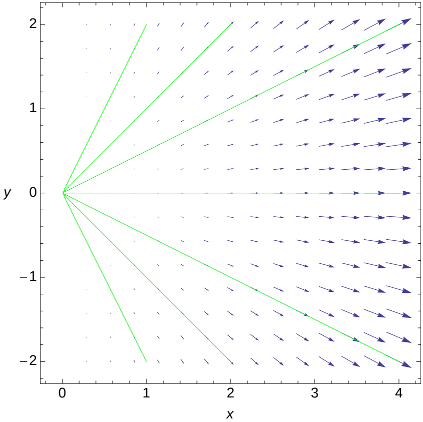

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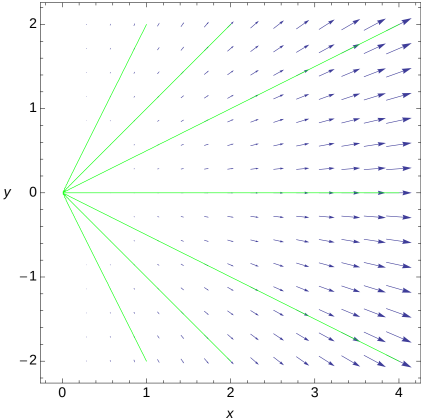

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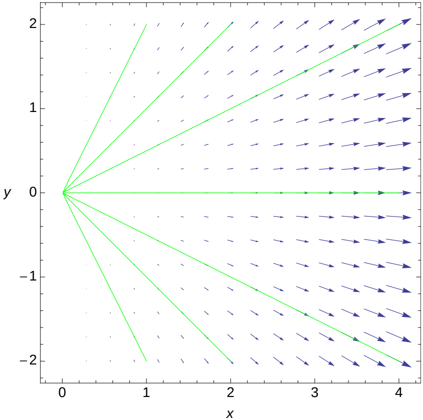

- equivalent to choosing $r(x, y)$ so that the derivative in the direction of $\vec{X}$ is 0:

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- choose $s(x, y)$ so that $s =$ constant curves are nowhere tangent to $r =$ constant curves and the derivative in the direction of $\vec{X}$ is uniform:

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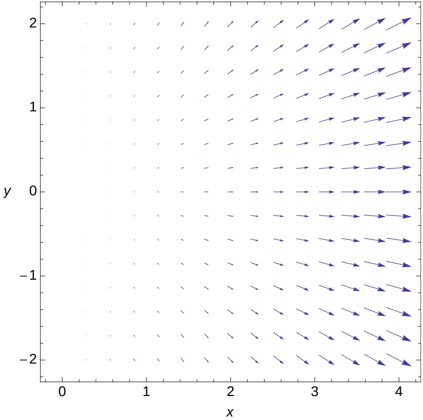

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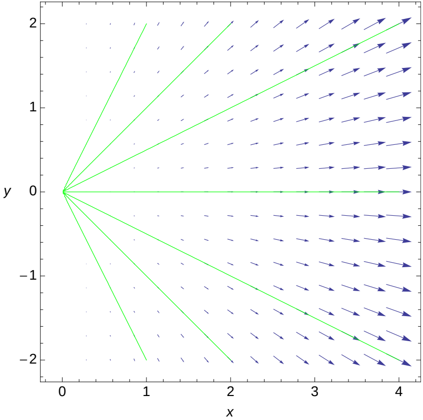

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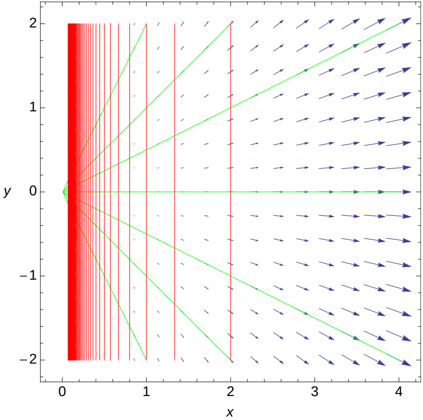

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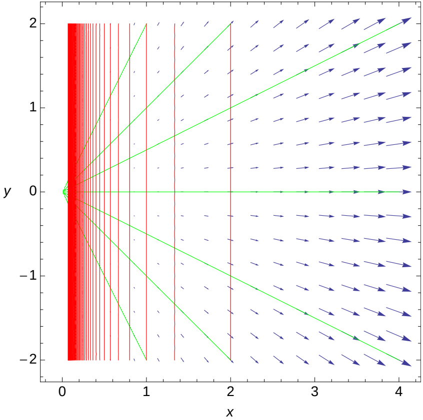

- looks intimidating but not so bad in practice since need only a specific solution rather than the general solution

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- these are the "magic" coordinates we used at the start

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- first-order *linear* PDE for $\xi(x, y)$ and $\eta(x, y)$

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  - try $\xi = ax + by + c$ and $\eta = \alpha x + \beta y + \gamma$

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- Example: $\dfrac{dy}{dx} = y$ so $f(x, y) = y$
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  - so $\vec{X} = (1, 0)$ and $\vec{X} = (0, y)$ are symmetry vector fields

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A few references

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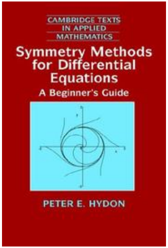
Peter Hydon, *Symmetry Methods for Differential Equations: A Beginner's Guide*, Cambridge, 2000.



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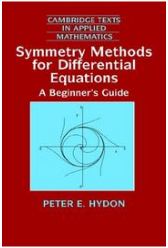

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