Name ____________________

MATH 301 | Differential Equations | Spring 2011 | Exam #4

Instructions: All of the work you submit must be the product of your own independent efforts. You can consult your class notes and the course text. You can refer to texts from prerequisite courses (the calculus sequence and linear algebra) as needed for reference on background skills and ideas. You can also use relevant computing technology (for computations and graphics but not as a reference). Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me. You can consult with me to clarify exam questions. Please do so right away if you suspect there is a significant typographical error.

This exam is a tool to help me (and you) assess how well you have learned the course material. As such, you should report enough written detail for me to understand how you reach conclusions and results in each problem.

The exam is due by 2 pm on Wednesday, May 11. You can bring the exam to my office anytime before 2 pm. If I am not in, you can either slide your exam under the door or give your exam to our department secretary in TH 414.

Each of the four problems has a value of 25 points for 100 points total.

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1. Use equilibrium point and nullcline analysis to make a phase portrait for the system

$$\frac{dx}{dt} = 4x - 2x^2 - xy$$
$$\frac{dy}{dt} = xy - 3y$$

2. Consider the system

$$\frac{dx}{dt} = x^2 + 3x - 2xy$$
$$\frac{dy}{dt} = y^2 - 3y - 2xy$$

   (a) Find the four equilibrium points for this system.

   (b) Show that this is a Hamiltonian system and find a Hamiltonian.

   (c) Use linearization and the Hamiltonian to show that one of the equilibrium points is a center and the other three are saddles.

   (d) Describe each of the different categories of solution curve for this system.

3. Consider the system

$$\frac{dx}{dt} = -x^3 - x^2 - 2xy$$
$$\frac{dy}{dt} = 2x^3y - x - 1$$

   (a) Find the equilibrium points for this system.

   (b) Show that $x = 0$ is a solution curve.

   (c) Show that $L(x, y) = xe^x e^{-y^2}$ is a Lyapunov function for each of the two regions $x < 0$ and $x > 0$.

   (d) Use this Lyapunov function to develop at least one interesting claim about the phase portrait for this system.

4. Here is a simple model for the spread of a disease in an isolated population. Let T represent the total number of individuals, S represent the number of susceptible individuals, I represent the number of infected individuals, and R represent the number of recovered individuals. Thus $T = S + I + R$. Assume a recovered individual has immunity and cannot be reinfected. Also, assume the disease is not fatal and spreads on a time scale that is short compared with the life span of individuals so we can assume T is constant. The basic assumptions in the model are

- infection occurs due to interaction between susceptible and infected individuals
- infected individuals recover at a constant percentage rate

We can incorporate these assumptions into a simple model as

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SI \\ \frac{dI}{dt} &= \alpha SI - \beta I \\ \frac{dR}{dt} &= \beta I\end{aligned}$$

where α and β are positive constants. Note that we really need only consider the first two equations since these determine S and I and we can then compute $R = T - S - I$. Also, we need only consider the first quadrant of the SI-plane.

(a) Explain the connection between the terms $-\alpha SI$ in the first equation and αSI in the second equation. Also, explain the connection between the terms $-\beta I$ in the second equation and βI in the third equation.

(b) Find all equilibrium points in the SI-plane.

(c) Show that linearization at any of these equilibrium points is not useful.

(d) Show that the quantity $\alpha I + \alpha S - \beta \ln S$ is constant along solution curves of the system.

(e) Make a phase portrait for the first quadrant of the SI-plane.

(f) Based on your phase portrait, give a qualitative description of how the disease spreads in a situation that starts with one infected individual amid a large number of susceptible individuals so that the initial point is a small perturbation away from $(T, 0)$.

(g) For the situation in (f), consider the specific parameter values $T = 3000$, $\alpha = 0.00004$, and $\beta = 0.05$. Add quantitative details to your description of how the disease spreads.

(h) Continuing from (g), suppose we can cut the transmission rate in half to $\alpha = 0.00002$. Does this produce a significant change in the spread of the disease (either qualitatively or quantitatively)?