

**Instructions:** You can work on the problems in any order. Clearly number your work for each problem. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (100 points total)

---

1. Compute the Laplace transform of  $f(t) = t$  directly from the definition.

Hint: Integration by parts formula:  $\int u dv = uv - \int v du$ . (6 points)

2. Find the inverse Laplace transform of  $\frac{e^{-2s}}{s-3}$ . (4 points)

3. Use the Laplace transform method to solve the initial value problem

$$\frac{dy}{dt} + 4y = \sin(3t) \quad y(0) = 2$$

(10 points)

4. Consider the differential equation  $\frac{dy}{dt} = 9y - y^3$ .

(a) List all the categories we've used in our course that apply to this differential equation. (3 points)

(b) Sketch a slope field for this differential equation using a window that shows all of the main features. (5 points)

(c) On your slope field from (b), sketch the specific solution satisfying  $y(0) = 1$ . Describe the asymptotic behaviors of this solution for  $t \rightarrow -\infty$  and  $t \rightarrow \infty$ . (4 points)

5. Consider the differential equation  $\frac{dy}{dt} = 4t^3y^2$ .

(a) List all the categories we've used in our course that apply to this differential equation. (3 points)

(b) Find the general solution. (8 points)

(c) Find the specific solution satisfying  $y(2) = 5$ . (3 points)

6. Consider the differential equation  $\frac{dy}{dt} = 2y + te^{-3t}$ .

(a) List all the categories we've used in our course that apply to this differential equation. (3 points)

(b) Describe the algebraic structure of the general solution. (3 points)

(c) Find the general solution. (9 points)

7. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -5x + 6y \\ \frac{dy}{dt} &= -3x + y\end{aligned}$$

- (a) Show where this system lies on the trace-determinant plane and give the nature of the phase portrait (with either a sketch or a label). (5 points)
- (b) Find the general solution. (9 points)

8. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\pi}x \sin(2\pi y) \\ \frac{dy}{dt} &= xy - 4y^2\end{aligned}$$

- (a) Show that  $(4, 1)$  is an equilibrium point. (Note that you do not need to find all equilibrium points. Just show that  $(4, 1)$  is an equilibrium point.) (3 points)
- (b) Determine what linearization says about the phase portrait of the system near  $(4, 1)$ . (10 points)

9. Do *one* of the following two problems. Circle the problem number of the one you submit. (12 points)

- (A) A damped harmonic oscillator is modeled by

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0$$

where  $m$ ,  $b$ , and  $k$  are positive constants. Show how to find the condition involving these parameters that marks the transition between underdamped motion and overdamped motion.

- (B) A population for a particular species with logistic internal growth and a constant harvesting rate is modeled by

$$\frac{dp}{dt} = kp \left(1 - \frac{p}{M}\right) - H$$

where  $k$ ,  $M$ , and  $H$  are positive constants. Consider a scenario in which we can control the value of  $H$  but the values of  $k$  and  $M$  are fixed. Find the condition that  $H$  must satisfy in order to guarantee the existence of a positive equilibrium value for  $p$ .