Name ______________________________

MATH 301 Differential Equations Spring 2009 Exam #3

Instructions: You can work on the problems in any order. Clearly number your work for each problem. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (100 points total)

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1. Consider the one-parameter family of systems

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} a & a \\ 1 & -a \end{bmatrix} \vec{Y}$$

where a can be any real number. Determine all possible phase portrait types for this family. For each possible phase portrait type, give the corresponding value(s) of a. (14 points)

2. Consider the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0.$$

(a) Find the specific solution satisfying the initial conditions $y(0) = 4$ and $y'(0) = 0$. (15 points)

(b) Sketch a plot of your solution from (a) and describe the long-term behavior of the solution. (5 points)

3. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4t + 5e^{-4t}.$$

(15 points)

4. Consider the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3e^{-2t}.$$

(a) Explain why there are no particular solutions of the form $y_p = Ae^{-2t}$. (4 points)

(b) Find a particular solution. (8 points)

There is more on the flip side.

5. Consider the damped harmonic oscillator model

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

with $m > 0$, $b \geq 0$, and $k > 0$. For each of the following parameter value sets, determine if the motion is underdamped, critically damped, or overdamped. Also sketch a representative phase portrait in the yv-plane. Your phase portraits should be qualitatively correct but need not be quantitatively correct. (8 points each)

(a) $m = 2,\ b = 4,\ k = 5$

(b) $m = 2,\ b = 5,\ k = 3$

6. Analyze the long-term behavior of the damped harmonic oscillator with external forcing modeled by

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 10y = 2\sin(3t)$$

(14 points)

7. Show that

$$y(t) = c_1\frac{1}{t^2} + c_2t^3 + t^3\ln t$$

is the general solution of the differential equation

$$t^2\frac{d^2y}{dt^2} - 6y = 5t^3$$

for the interval $(0, \infty)$.

Note: You do not need to show where this formula for $y(t)$ comes from. (9 points)