Name ______________________________

MATH 301 Differential Equations Spring 2009 Exam #2

Instructions: You can work on the problems in any order. Clearly number your work for each problem. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (100 points total)

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1. Consider the system

$$\frac{dx}{dt} = x^2 y + y^2$$
$$\frac{dy}{dt} = x + xy$$

   (a) Make a plot of the xy-plane showing the direction vectors for this system along the x-axis and along the y-axis. (6 points)

   (b) Consider the solution curve for an initial condition $x = x_0 > 0$ and $y = y_0 > 0$ at $t = 0$. Based on your plot from (a), what can you saw about this solution curve for $t > 0$? (4 points)

   (c) Compute two steps of Euler's method using a step size of $\Delta t = 0.01$ for the initial condition $x = 1$ and $y = 2$ at $t = 0$. (7 points)

2. Consider two species that interact in a mutually beneficial way. The first species can thrive in the absence of the second species. The second species has a negative growth rate in the absence of the first. (As an example, the first could be a species of bee and the second could be a plant requiring the bee species for pollination. The plant's nectar is a food source for the bee but not the only food source. The bee is the only pollinator for the plant.) Let x measure the amount of the first species in a fixed territory and y measure the amount of the second species in that same territory. A very simple model for the rates of change is

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{N}\right) + \beta xy$$
$$\frac{dy}{dt} = -\gamma y + \delta xy$$

   where α, β, γ, δ, and N are positive constants.

   (a) Explain how each term on the right sides of these equations relates to the assumptions stated above. (6 points)

   Note: For the remaining parts of this problem, use the values $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 0.5$, and $N = 1$. A phase portrait for these parameter values is given on the accompanying sheet.

   (b) Find any equilibrium points for this model. (6 points)

   (c) Describe what the model predicts for the two species if the initial condition is $x = 0.5$ and $y = 1.0$ for $t = 0$. Your description can include graphs. (5 points)

   (d) Describe what the model predicts for the two species if the initial condition is $x = 0.5$ and $y = 3.0$ for $t = 0$. Your description can include graphs. (5 points)

There is more on the flip side.

3. Find the general solution of the system

$$\frac{dx}{dt} = x + 6y$$
$$\frac{dy}{dt} = 2x$$

Note: Include details on how you find any relevant eigenvalues and eigenvectors. (15 points)

Note: For the remaining problems, you should be able to find the relevant eigenvalues and eigenvectors on the accompanying "catalog of matrices".

4. Consider the system

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \quad \text{with} \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

   (a) Find the general solution for this system. (9 points)

   (b) Sketch a phase portrait for this system. (7 points)

   (c) Find the specific solution for the initial condition $\vec{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (6 points)

5. Consider the system

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{Y}$$

   (a) Find the general solution for this system. (9 points)

   (b) Sketch a phase portrait for this system. (7 points)

6. Suppose

- λ is a repeated eigenvalue of the real constant (2×2) matrix A,
- $\vec{w}_1$ is an eigenvector of A for λ, and
- $\vec{w}_0$ is a solution of $(A - \lambda I)\vec{w}_0 = \vec{w}_1$.

Show that $e^{\lambda t}(\vec{w}_0 + t\vec{w}_1)$ is a solution of $\frac{d\vec{Y}}{dt} = A\vec{Y}$. (8 points)