Some theory on linear homogeneous ODES

We'll use the following notation:

- $C^n(a,b)$ is the vector space of functions with continuous n^{th} derivative on the domain (a,b).
- $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$ where each of the coefficients a_i is a function of the independent variable and D is the differentiation operator
- $W_S(t)$ is the Wronskian of the set $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ defined as

$$W_S(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \ldots & f_n(t) \\ f_1'(t) & f_2'(t) & \ldots & f_n'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \ldots & f_n^{(n-1)}(t) \end{bmatrix}$$

Theorem 1. *Let $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ be a set of functions in $C^n(a,b)$. If there is a t_0 in (a,b) such that $W_S(t_0)$ is nonzero, then S is linearly independent.*

Proof. Start with the defining equation of linear independence

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = \theta(t)$$

where $\theta(t)$ is the zero function. We must show that the only solution is the trivial solution. Differentiate both sides of this equation $n-1$ times to generate a system of equations

$$\begin{aligned} c_1 f_1(t) + c_2 f_2(t) &+ \cdots + c_n f_n(t) &= \theta(t) \\ c_1 f_1'(t) + c_2 f_2'(t) &+ \cdots + c_n f_n'(t) &= \theta(t) \\ \vdots \qquad\quad & \qquad \vdots & \vdots \\ c_1 f_1^{(n-1)}(t) + c_2 f_2^{(n-1)}(t) &+ \cdots + c_n f_n^{(n-1)}(t) &= \theta(t) \end{aligned}$$

The Wronskian $W_S(t)$ is defined as the determinant of the coefficient matrix for this system. Hence, if the Wronskian is nonzero for t_0 in (a,b), the system has a unique solution for that value t_0. This unique solution must be the trivial solution because the system of equations is homogeneous. Thus, the trivial solution is the only solution for all values of t. □

We now look at the set of solutions for an n^{th} order, linear homogeneous differential equation $L[y(t)] = \theta(t)$. We can view the solution set as the null space $\mathcal{N}(L)$, defined as

$$\mathcal{N}(L) = \{y \in C^n(a,b) | L[y] = 0\}.$$

Theorem 2. *If $a_{n-1}(t), \ldots, a_1(t), a_0(t)$ are continuous for all t in (a,b) and L is defined as above, then the solution set $\mathcal{N}(L)$ is a subspace of $C^n(a,b)$ of dimension n.*

Proof. Since L is a linear transformation, we know that $\mathcal{N}(L)$ is a subspace of $C^n(a,b)$ by a standard theorem of linear algebra (for example, see Theorem NSLTS of FCLA). To show that it has dimension n, we will find a basis with n elements.

To begin, we claim the existence of n solutions to the O.D.E. by the existence-uniqueness theorem. In particular, pick some t_0 in I and let $h_1(t)$, $h_2(t)$, $\ldots$, $h_n(t)$ be the solutions that satisfy the following sets of initial conditions

$$\begin{array}{llll} h_1(t_0)=1, & h_1'(t_0)=0, & \ldots, & h_1^{(n-1)}(t_0)=0 \\ h_2(t_0)=0, & h_2'(t_0)=1, & \ldots, & h_2^{(n-1)}(t_0)=0 \\ \vdots & \vdots & \vdots & \vdots \\ h_n(t_0)=0, & h_n'(t_0)=0, & \ldots, & h_n^{(n-1)}(t_0)=1 \end{array}$$

To prove that $\{h_1(t),h_2(t),\ldots,h_n(t)\}$ is a basis for $N(L)$, we must show two things: one, that the set is linearly independent; and two, that the set spans $N(L)$.

To show linear independence, we note that

$$W[h_1,h_2,\ldots,h_n](t_0)=1\neq 0.$$

By Theorem 1, the set $\{h_1(t),h_2(t),\ldots,h_n(t)\}$ is linearly independent.

To prove that the set $\{h_1(t),h_2(t),\ldots,h_n(t)\}$ spans $N(L)$, we must show that any other solution in $N(L)$ can be written as a linear combination of the elements in $\{h_1(t),h_2(t),\ldots,h_n(t)\}$. Let $y(t)$ be any solution. At t_0, this solution and its derivatives have some values

$$y(t_0)=c_1\ ,\quad y'(t_0)=c_2\ ,\quad \ldots\quad ,\quad y^{(n-1)}(t_0)=c_n.$$

Consider the solution given by the linear combination $c_1h_1(t)+c_2h_2(t)+\cdots+c_nh_n(t)$. Note that at t_0, this solution and its derivatives has the same values as the solution $y(t)$ and its derivatives. Hence, by the existence-uniqueness theorem, we have

$$y(t)=c_1h_1(t)+c_2h_2(t)+\cdots+c_nh_n(t).$$

This gives $y(t)$ as a linear combination of the elements in $\{h_1(t),h_2(t),\ldots,h_n(t)\}$ and thus completes the proof. □

Theorem 3. *Let $S=\{y_1(t),y_2(t),\ldots,y_n(t)\}$ be a set of n solutions to the n^{th} order linear differential equation $L[y]=0$ with coefficient functions a_i that are continuous for (a,b). The set S is linearly independent if and only if there is a t_0 in (a,b) such that $W_S(t_0)$ is nonzero.*

Proof. The proof of one direction follows immediately from Theorem 1. The proof of the other direction is an exercise. □

Exercises

1. Determine if $S=\{t^3,|t|^3\}$ is linearly independent in $C^2(-\infty,\infty)$ without using the Wronskian. Now compute the Wronskian of S. Comment on these results in relation to Theorems 1 and 3.

2. Finish the proof of Theorem 3. Hint: Work with the contrapositive of the statement to be proven: If $W_S(t)=0$ for all t in (a,b), then S is linearly dependent. Don't forget that here the set S consists of solutions to $L[y]=0$.