What is a solution?

Definition:
A solution of the initial value problem

$$\frac{dy}{dt} = f(t, y) \qquad \text{with } y = y_0 \text{ for } t = t_0$$

is a differentiable function defined on an interval (α, β) containing t_0 that satisfies both the differential equation and the initial condition.

Note:
To distinguish between the unknown y in the problem statement and a specific candidate for this unknown, we might write $y = \phi(t)$ and then require

$$\phi'(t) = f(t, \phi(t)) \quad \text{for all } t \text{ in } (\alpha, \beta) \qquad \text{with} \qquad \phi(t_0) = y_0.$$

What is a solution?

Example:

$$\frac{dy}{dt} = y \qquad \text{with } y = 5 \text{ for } t = 0$$

- One solution is $y = 5e^t$ for t in $(-1, 1)$.
- Another solution is $y = 5e^t$ for t in $(-\infty, \infty)$.
- We say that this second solution is an *extension* of the first solution.
- In fact, it is the *maximal extension* since the domain cannot be extended further.

An existence-uniqueness theorem: informal

Theorem (informal):
If f and $\partial f/\partial y$ are continuous for all points in the *ty*-plane near (t_0, y_0), then there is a unique solution to the initial value problem

$$\frac{dy}{dt} = f(t, y) \qquad \text{with} \qquad y(t_0) = y_0.$$

Notes:

- Need to be clear on what "points *near* (t_0, y_0)" means.
- Need to be clear on domain of the solution.
- For existence, need only continuity of f. For uniqueness, need continuity of both f and $\partial f/\partial y$.

An existence-uniqueness theorem: precise

Theorem:
If f and $\partial f/\partial y$ are continuous in a rectangle $\{(t, y) \big| a < t < b, c < y < d\}$ containing (t_0, y_0), then there is a value $\epsilon > 0$ defining an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for which there is a unique solution to the initial value problem

$$\frac{dy}{dt} = f(t, y) \qquad \text{with} \qquad y(t_0) = y_0.$$

Note:

- Uniqueness means that if y_1 and y_2 are functions defined for $(t_0 - \epsilon, t_0 + \epsilon)$ and each satisfies the IVP, then

$$y_1(t) = y_2(t) \quad \text{for all } t \text{ in } (t_0 - \epsilon, t_0 + \epsilon).$$

- No guarantee about solution or uniqueness extending beyond the interval $(t_0 - \epsilon, t_0 + \epsilon)$.