Name ______________________________

MATH 280 Multivariate Calculus Spring 2007 Exam #5

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (100 points total)

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1. (a) For a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and an oriented curve C in the domain of $\vec{F}$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is defined as the limit of a certain Riemann sum. Explain what is being added in this Riemann sum. Include a picture to explain the geometric meaning of relevant things in your explanation. (7 points)

   (b) For a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ and a surface S in the domain of f, the surface integral $\iint_S f\, dA$ is defined as the limit of a certain Riemann sum. Explain what is being added in this Riemann sum. Include a picture to explain the geometric meaning of relevant things in your explanation. (7 points)

2. Charge is distributed on a right circular cylinder of radius R and height H. (The cylinder is a surface, not a solid region. Also, there are no "caps" on the cylinder to worry about.) The area charge density is proportional to the distance from one end of the cylinder with a value of 0 at one end and a value of σ_0 at the other end. Compute the total charge on the region in terms of R, H, and σ_0. (12 points)

3. Set up a definite integral equal to the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xz^2\,\hat{\imath} + \hat{\jmath} + \sin x\,\hat{k}$ and C is the curve parametrized by $\vec{r}(t) = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + 5t\,\hat{k}$ for t from 6 to 2. Express the definite integral entirely in terms of one variable. You do not need to evaluate the integral. (12 points)

4. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = z^2\,\hat{\imath} + (z+1)\,\hat{\jmath} + (2xz + y)\,\hat{k}$ and C is a curve starting at $(3, 0, 2)$ and ending at $(1, 1, 1)$. (12 points)

5. Prove that the curl of the gradient of a scalar field is zero. That is, prove the identity $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ for a function $f : \mathbb{R}^3 \to \mathbb{R}$. (10 points)

There is more on the flip side.

6. Consider the vector field $\vec{F} = (x + \sin y)\,\hat{\imath} + y^2 z\,\hat{\jmath} + x^2\,\hat{k}$.

   (a) Compute the divergence of $\vec{F}$ for the point $(2, -3, 1)$. (7 points)

   (b) Consider $\vec{F}$ as the velocity field for fluid flow. Imagine a small drop of dye placed at the point $(2, -3, 1)$. Describe how the volume of the drop will change (instantaneously) as the dye particles move with the flow. (3 points)

   (c) Compute the curl of $\vec{F}$ for the point $(2, -3, 1)$. (7 points)

   (d) Consider $\vec{F}$ as the velocity field for fluid flow. Imagine a small paddlewheel placed at the point $(2, -3, 1)$. Compare the rotation of the paddlewheel when its axis is in the $\hat{\imath}$-direction with the rotation of the paddlewheel when its axis is in the $\hat{k}$-direction. (3 points)

7. Consider the surface integral $\displaystyle\oiint_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = yz\,\hat{\imath} + xy\,\hat{\jmath} + \sin(xy)\,\hat{k}$ and S is the boundary of the solid cube defined by $0 \le x \le 3$, $0 \le y \le 3$, and $0 \le z \le 3$. (Note that S itself is a surface and not a solid region.) Orient S with outward pointing normals.

   (a) Use the Divergence Theorem to give a triple integral that is equal to $\displaystyle\oiint_S \vec{F} \cdot d\vec{A}$. (8 points)

   (b) Evaluate $\displaystyle\oiint_S \vec{F} \cdot d\vec{A}$ by evaluating the triple integral you find in (a). (6 points)

8. Let S_1 be the upper hemisphere $x^2+y^2+z^2 = 1$ with $z \ge 0$. Let S_2 be the lower hemisphere $x^2 + y^2 + z^2 = 1$ with $z \le 0$. Orient both S_1 and S_2 with normal vectors pointing away from the origin. Let $\vec{F}$ be any vector field for which Stokes' Theorem applies on both S_1 and S_2. Use Stokes' Theorem to show that $\displaystyle\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}$ and $\displaystyle\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}$ must be opposites. (6 points)

Conclusions of the Fundamental Theorems

FT of Calculus: $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

FT for Line Integrals: $\displaystyle\int_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A)$

Stokes' Theorem: $\displaystyle\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}$

Divergence Theorem: $\displaystyle\iiint_R (\vec{\nabla} \cdot \vec{F})\,dV = \oiint_S \vec{F} \cdot d\vec{A}$