Solutions: Integrating a vector field over a surface

4. Compute $\displaystyle\iint_S \vec{F}\cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and S is the open right circular cylinder of radius 2 and height 6 centered at the origin with axis along the z-axis oriented so that area vectors point outward (i.e., away from the z-axis).

Solution:

In cylindrical coordinates, the cylinder is described by $r = 2$ for $0 \le \theta \le 2\pi$ and $-3 \le z \le 3$. Expressing cartesian coordinates in terms of cylindrical coordinates (with $r = 2$), we have

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = z.$$

Let $d\vec{r}_1$ be an infinitesmial displacement with z held constant so $dz = 0$ and thus

$$d\vec{r}_1 = \left(-2\sin\theta\,\hat{\imath} + 2\cos\theta\,\hat{\jmath} + 0\,\hat{k}\right)d\theta.$$

Let $d\vec{r}_2$ be an infinitesmial displacement with θ held constant so $d\theta = 0$ and thus

$$d\vec{r}_2 = \left(0\,\hat{\imath} + 0\,\hat{\jmath} + \hat{k}\right)dz.$$

Now compute

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = \left(2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + 0\,\hat{k}\right)d\theta dz.$$

Note that we can match this result with our geometric intuition that all area element vectors $d\vec{A}$ on this cylinder are horizontal. Along the surface, we have

$$\vec{F} = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + z\,\hat{k}$$

so

$$\vec{F}\cdot d\vec{A} = 4\,d\theta dz.$$

Putting together these pieces, we get

$$\iint_S \vec{F}\cdot d\vec{A} = \int_{-3}^{3}\int_0^{2\pi} 4\,d\theta dz = 4\int_{-3}^{3} dz \int_0^{2\pi} d\theta = 4(6)(2\pi) = 48\pi.$$

5. Compute $\displaystyle\iint_S \vec{F}\cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and S is the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$ oriented so that area vectors point outward (i.e., away from the z-axis).

Solution:

In cylindrical coordinates, the equation of the paraboloid is $z = r^2$. The piece of the paraboloid with $0 \le z \le 1$ projects onto the disk of radius 1 centered at the origin in the $z = 0$ plane so we have $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Expressing cartesian coordinates in terms of cylindrical coordinates, we have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = r^2.$$

Let $d\vec{r}_1$ be an infinitesmial displacement with r held constant so $dr = 0$ and thus

$$d\vec{r}_1 = \left(-r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\,\hat{k}\right)d\theta.$$

Let $d\vec{r}_2$ be an infinitesmial displacement with θ held constant so $d\theta = 0$ and thus

$$d\vec{r}_2 = \left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 2r\,\hat{k}\right)dr.$$

Now compute

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = \left(2r^2\cos\theta\,\hat{\imath} + 2r^2\sin\theta\,\hat{\jmath} - r\,\hat{k}\right)dr d\theta.$$

Note that we can match this result with our geometric intuition that area element vectors $d\vec{A}$ on the paraboloid that are pointing away from the z-axis will have negative $\hat{k}$-components. Along the surface, we have

$$\vec{F} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^2\,\hat{k}$$

so

$$\vec{F}\cdot d\vec{A} = \left(2r^3\cos^2\theta + 2r^3\sin^2\theta - r^3\right)dr d\theta = r^3\,dr d\theta.$$

Putting together these pieces, we get

$$\iint\limits_S \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^1 r^3\,dr d\theta = \int_0^{2\pi} d\theta \int_0^1 r^3 dr = (2\pi)\left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

We could also approach this problem using cartesian coordinates. From $z = x^2 + y^2$, we get $dz = 2x\,dx + 2y\,dy$. Let $d\vec{r}_1$ be an infinitesmial displacement with x held constant so $dx = 0$ and thus

$$d\vec{r}_1 = \left(0\,\hat{\imath} + \hat{\jmath} + 2y\,\hat{k}\right)dy.$$

Let $d\vec{r}_2$ be an infinitesmial displacement with y held constant so $dy = 0$ and thus

$$d\vec{r}_2 = \left(\hat{\imath} + 0\,\hat{\jmath} + 2x\,\hat{k}\right)dx.$$

Now compute

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = \left(2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}\right)dx dy.$$

Along the surface, we have

$$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + (x^2 + y^2)\,\hat{k}$$

so

$$\vec{F} \cdot d\vec{A} = \left(2x^2 + 2y^2 - x^2 - y^2\right) dxdy = \left(x^2 + y^2\right) dxdy.$$

In cartesian coordinates, the unit disk in the $z = 0$ plane onto which the surface projects is described by $-1 \leq x \leq 1$ and $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Putting together these pieces, we get

$$\iint\limits_S \vec{F} \cdot d\vec{A} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2 + y^2\right) dydx.$$

Perhaps the easiest way to evaluate the iterated integral is to transform to polar coordinates, giving us

$$\iint\limits_S \vec{F} \cdot d\vec{A} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2 + y^2\right) dydx. = \int_0^{2\pi} \int_0^1 r^2\, r dr d\theta$$

Note that this is exactly the iterated integral we got using the cylindrical coordinates approach. So, the details of evaluating it are identical and we get

$$\iint\limits_S \vec{F} \cdot d\vec{A} = \frac{\pi}{2}.$$