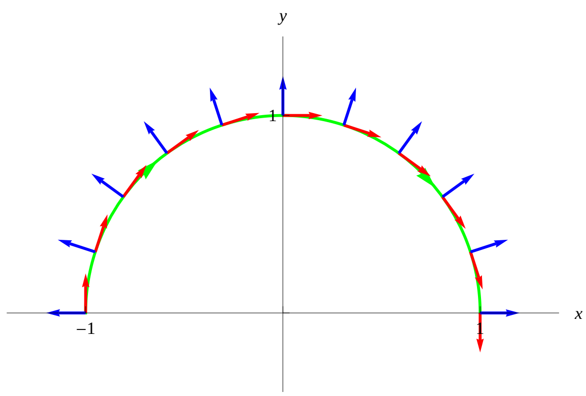
Vector curve integral problems

For each of the following,

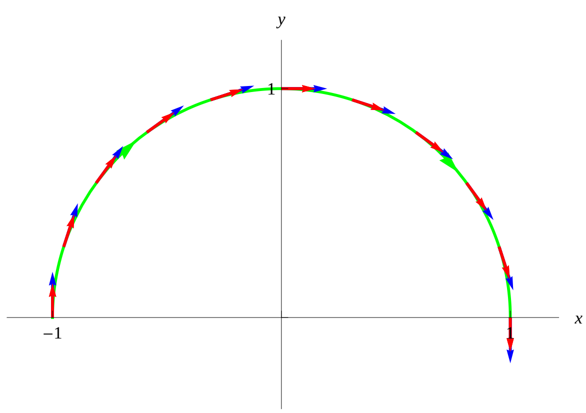
- Sketch the given vector field $\vec{F}$ and the given curve C.
- Use your sketch to determine or estimate the sign of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.
- Compute the value of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

1. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \geq 0$



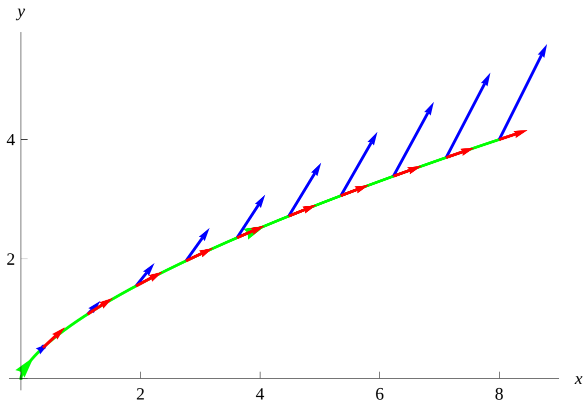
Answer: 0

2. $\vec{F} = y\,\hat{\imath} - x\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \geq 0$



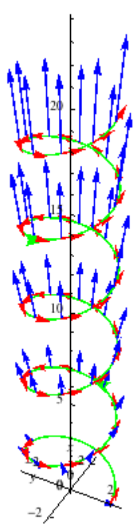
Answer: π

3. $\vec{F} = y\,\hat{\imath} + x\,\hat{\jmath}$
   C is given by $x = t^3$ and $y = t^2$ for $t = 0$ to $t = 2$



Answer: 32

4. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$
   C is the helix with constant pitch wrapping 5 times around a (right circular) cylinder of radius 2 and height 20



Answer: 200

Solution:

We can describe the helix using cylindrical coordinates with $r = 2$ to get

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = \frac{20}{10\pi}\theta = \frac{2}{\pi}\theta$$

for $0 \leq \theta \leq 2\pi$. Note that the helix having constant pitch means that z is proportional to *theta*; the proportionality constant is determined by the requirement that helix goes up 20 units in 5 wraps. From these, we compute

$$dx = -2\sin\theta\,d\theta \qquad dy = 2\cos\theta\,d\theta \qquad dz = \frac{2}{\pi}\,d\theta$$

to get

$$d\vec{r} = \left(-2\sin\theta\,\hat{\imath} + 2\cos\theta\,\hat{\jmath} + \frac{2}{\pi}\,\hat{k}\right)d\theta.$$

Along the curve, the vector field is

$$\vec{F} = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + \frac{2}{\pi}\theta\,\hat{k}.$$

Dotting these together gives us

$$\vec{F}\cdot d\vec{r} = \left(-4\sin\theta\cos\theta + 4\cos\theta\sin\theta + \frac{4}{\pi^2}\theta\right)d\theta = \frac{4}{\pi^2}\theta\,d\theta.$$

Putting together the details, we get

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{10\pi} \frac{4}{\pi^2}\theta\,d\theta = \frac{4}{\pi^2}\frac{\theta^2}{2}\bigg|_0^{10\pi} = 200.$$