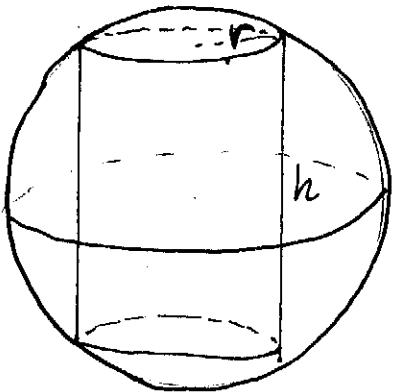
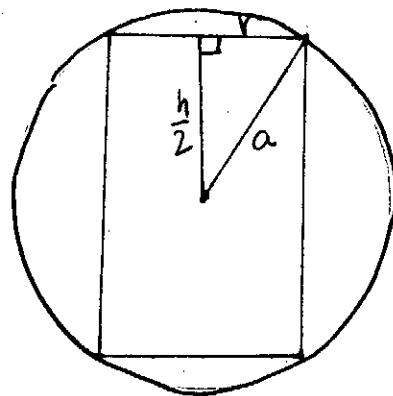


Section 12.8 #10Approach #1 : no coordinates

perspective view:



cross-section

objective: maximize $A = 2\pi rh$ constraint: $r^2 + \frac{1}{4}h^2 = a^2$ (by Pythagorean thm in cross-section)

$$\vec{\nabla} A = 2\pi h \hat{r} + 2\pi r \hat{h}$$

$$\vec{\nabla} g = 2r \hat{r} + \frac{1}{2}h \hat{h}$$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \Rightarrow 2\pi h = \lambda \cdot 2r \quad (1)$$

$$2\pi r = \lambda \cdot \frac{1}{2}h \quad (2)$$

$$r^2 + \frac{1}{4}h^2 = a^2 \quad (3)$$

$$(1) \Rightarrow \lambda = \frac{\pi h}{r}$$

$$(2) \Rightarrow \lambda = \frac{4\pi r}{h}$$

$$\left. \begin{array}{l} (1) \Rightarrow \lambda = \frac{\pi h}{r} \\ (2) \Rightarrow \lambda = \frac{4\pi r}{h} \end{array} \right\} \Rightarrow \frac{\pi h}{r} = \frac{4\pi r}{h} \Rightarrow h^2 = 4r^2$$

Substitute into (3):

$$r^2 + \frac{1}{4}(4r^2) = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}$$

$$\Rightarrow h = 2r = \sqrt{2}a$$

Maximum surface area:

$$A = 2\pi \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{2}a = 2\pi a^2$$

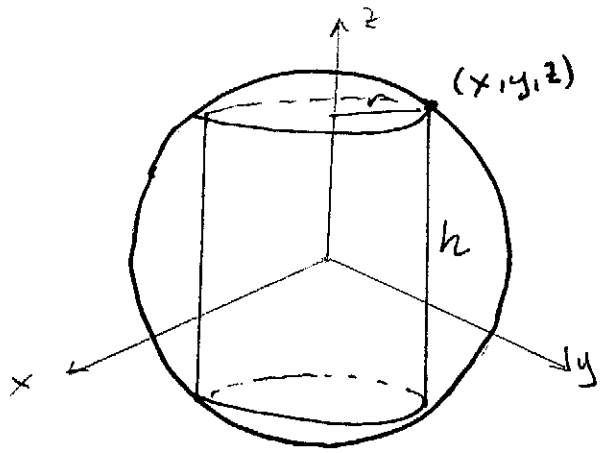
Approach #2: with coordinates

$$\text{cylinder: } x^2 + y^2 = r^2$$

$$\text{sphere: } x^2 + y^2 + z^2 = a^2$$

$$h = 2z \quad r^2 = x^2 + y^2$$

where (x, y, z) is a point where cylinder & sphere intersect



$$\text{objective: maximize } A = 2\pi r h = 2\pi \sqrt{x^2 + y^2} \cdot 2z = 4\pi z \sqrt{x^2 + y^2}$$

$$\text{constraint: } x^2 + y^2 + z^2 = a^2$$

$$\vec{\nabla} A = 4\pi \frac{xz}{\sqrt{x^2 + y^2}} \hat{i} + 4\pi \frac{yz}{\sqrt{x^2 + y^2}} \hat{j} + 4\pi \sqrt{x^2 + y^2} \hat{k}$$

$$\vec{\nabla} g = 2x \hat{i} + 2y \hat{j} + 2z \hat{k}$$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \Rightarrow 4\pi \frac{xz}{\sqrt{x^2 + y^2}} = \lambda \cdot 2x \quad (1)$$

$$4\pi \frac{yz}{\sqrt{x^2 + y^2}} = \lambda \cdot 2y \quad (2)$$

$$4\pi \sqrt{x^2 + y^2} = \lambda \cdot 2z \quad (3)$$

$$x^2 + y^2 + z^2 = a^2 \quad (4)$$

$$\left. \begin{array}{l} (1) \text{ or } (2) \Rightarrow \lambda = 2\pi \frac{z}{\sqrt{x^2 + y^2}} \\ (3) \Rightarrow \lambda = 2\pi \frac{\sqrt{x^2 + y^2}}{z} \end{array} \right\} \Rightarrow \frac{z}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2 + y^2}}{z} \Rightarrow z^2 = x^2 + y^2$$

Substitute z^2 for $x^2 + y^2$ in (4):

$$z^2 + z^2 = a^2 \Rightarrow 2z^2 = a^2 \Rightarrow z = \frac{a}{\sqrt{2}}$$

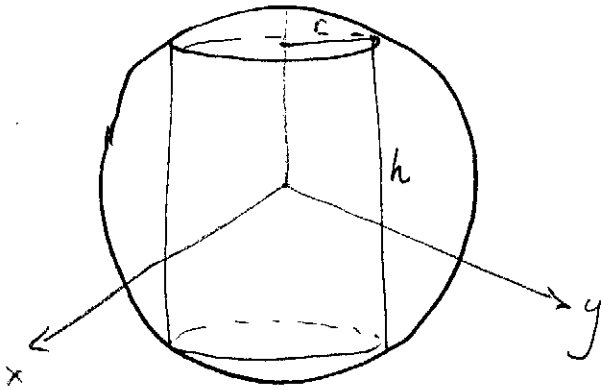
$$\text{So } h = 2z = 2 \frac{a}{\sqrt{2}} = \sqrt{2} a$$

$$r = \sqrt{x^2 + y^2} = z = \frac{a}{\sqrt{2}}$$

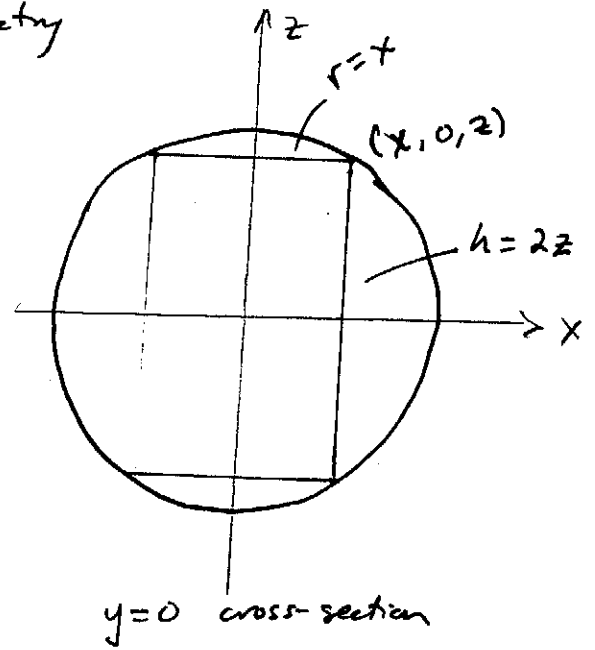
Maximum surface area:

$$A = 2\pi \frac{a}{\sqrt{2}} \cdot \sqrt{2} a = 2\pi a^2$$

Approach #3: with coordinates and symmetry



perspective



$y=0$ cross-section

cylinder: $x^2 + y^2 = r^2$

sphere: $x^2 + y^2 + z^2 = a^2$

with $y=0$:

$$x^2 = r^2 \Rightarrow x = r$$

$$x^2 + z^2 = a^2$$

Objective: maximize $A = 2\pi r h = 2\pi x(2z) = 4\pi x z$

constraint: $x^2 + z^2 = a^2$

$$\vec{\nabla} A = 4\pi z \hat{i} + 4\pi x \hat{j}$$

$$\vec{\nabla} g = 2x \hat{i} + 2z \hat{j}$$

$$\vec{\nabla} A = \lambda \vec{\nabla} g \Rightarrow 4\pi z = \lambda \cdot 2x \quad (1)$$

$$4\pi x = \lambda \cdot 2z \quad (2)$$

$$x^2 + z^2 = a^2 \quad (3)$$

$$(1) \Rightarrow \lambda = \frac{2\pi z}{x}$$

$$(2) \Rightarrow \lambda = \frac{2\pi x}{z}$$

$$\left. \begin{array}{l} \lambda = \frac{2\pi z}{x} \\ \lambda = \frac{2\pi x}{z} \end{array} \right\} \frac{2\pi z}{x} = \frac{2\pi x}{z} \Rightarrow z^2 = x^2$$

Substitute into (3): $x^2 + x^2 = a^2 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{2}}$

$$\Rightarrow z = \frac{a}{\sqrt{2}}$$

So $r = x = \frac{a}{\sqrt{2}}$ and $h = 2z = 2 \cdot \frac{a}{\sqrt{2}} = \sqrt{2} a$

Maximum surface area: $A = 2\pi r h = 2\pi \cdot \frac{a}{\sqrt{2}} \cdot \sqrt{2} a = 2\pi a^2$