A precise definition of limit

What, precisely, do we mean when we declare

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{or} \qquad \lim_{x\to 5}\frac{x^2-25}{x-5}=10 \qquad \text{or} \qquad \lim_{x\to 3}4x=12\ ?$$

Let's focus on the last of these for simplicity. Here, we are dealing with the function $f(x)=4x$ for x near the fixed value $a=3$. We declare that $\lim_{x\to 3}4x=12$ because we can get the outputs $f(x)=4x$ as close to 12 as requested for all inputs x as close to 3 as needed. That is, if someone issues a challenge to get the outputs of $f(x)=4x$ within 0.1 of 12, we can respond by showing that this happens for all inputs within 0.025 of 3. If the challenge is a smaller target around 12, we can respond with a smaller launch pad around 3 so that any input x from the launch pad generates an output $f(x)$ in the target.

The simple example of $\lim_{x\to 3}4x=12$ fails to illustrate one essential feature of limits: we never have to consider 3 itself as part of the launch pad. To better understand this, let's look at the statement $\lim_{x\to 5}\frac{x^2-25}{x-5}=10$. We say this statement is true because for any challenge of a target around 10, we can find a succesful launch pad around 5. For example, if the challenge is to get within 0.1 of 10, it works to use a launch pad of 0.1 on either side of 5. But, the launch pad does not include 5 itself because 5 is not in the domain of $f(x)=\frac{x^2-25}{x-5}$.

With these examples in mind, let's look at the general statement $\lim_{x\to x_0}f(x)=L$. Here's a definition of what this means.

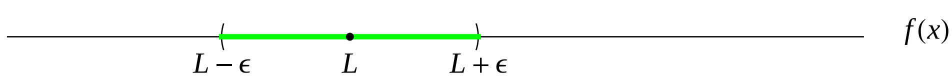
Definition (Version 1): The number L is the limit of f at x_0 if for each target around L, there is a successful launch pad around x_0.

To make this completely precise, we need to specify what we mean by

- a *target* around L,
- a *launch pad* around x_0, and
- a *successful* launch pad.

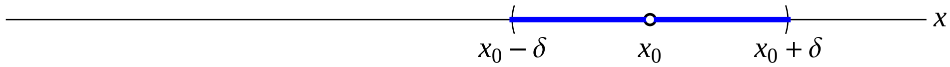
A *target* around L is simply an open interval centered at L. We'll typically use ε to denote the "radius" of this interval on either side of L. So, a target with radius ε is just the open interval from $L-\varepsilon$ to $L+\varepsilon$ as shown in the figure below. Note that we think of L and the target as being on the $f(x)$ axis. (We might choose to plot this as a vertical axis.) An output $f(x)$ is in this target if

$$L-\varepsilon<f(x)<L+\varepsilon \qquad \text{which is the same as} \qquad |f(x)-L|<\varepsilon.$$

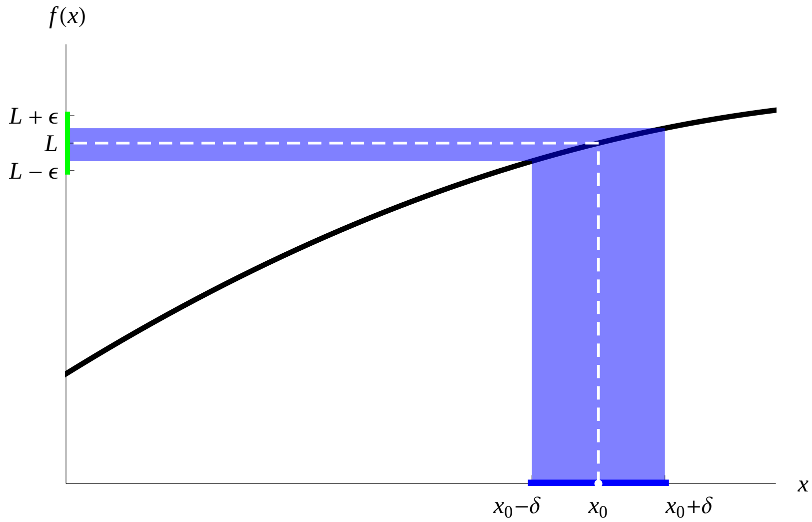


A *launch pad* around x_0 is simply an open interval centered at x_0 with x_0 taken out. We'll typically use δ to denote the "radius" of this interval on either side of x_0. So, a launch pad with radius δ is just the open interval from $x_0 - \delta$ to $x_0 + \delta$ with x_0 taken out as shown in the figure below. An input x is in this launch pad if

$$x_0 - \delta < x < x_0 + \delta \quad \text{and} \quad x \neq x_0 \qquad \text{which is the same as} \qquad 0 < |x - x_0| < \delta.$$



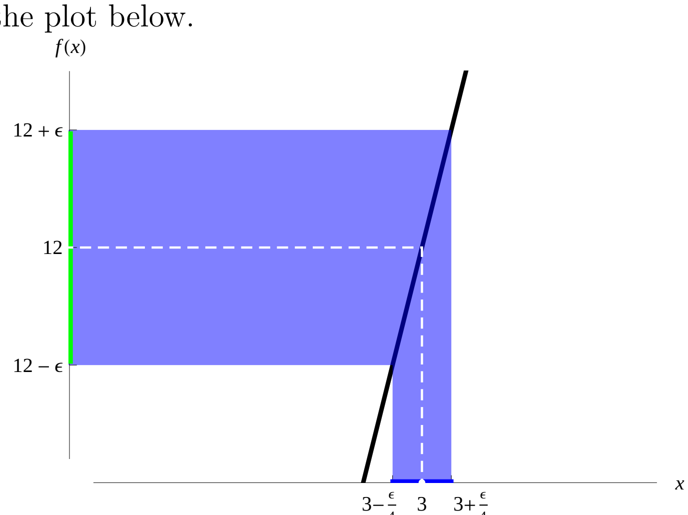
With f, x_0, and L specified, we can pick a target and then look at a launch pad. For a given target, a launch pad is *successful* if every input x in the launch pad has an output $f(x)$ in the target. The figure below illustrates a target (the interval shown on the vertical axis) and a successful launch pad (the interval shown on the horizonatal axis) for a generic function.



A launch pad is *not* successful if contains any input x for which the output $f(x)$ is not in the target.

Example 1

Consider the function $f(x) = 4x$ for $x_0 = 3$. Since this function scales all inputs by a factor of 4, the radius of a successful launch pad will need to be no bigger than 1/4 of the radius of a given target. For the target radius $\varepsilon = 0.1$, the launch pad with radius $\delta = 0.025$ is successful. (In fact, any launch pad with radius $\delta \leq 0.025$ is successful.) More generally, for the target radius ε, the launch pad with radius $\delta = \varepsilon/4$ is successful. This is illustrated in the plot below.



Let's demonstrate this algebraically. Suppose x is in the launch pad with radius $\delta = \varepsilon/4$. Then $x \neq 3$ and

$$3 - \frac{\varepsilon}{4} < x < 3 + \frac{\varepsilon}{4}.$$

Multiplying through by 4 gives us

$$4(3 - \frac{\varepsilon}{4}) < 4x < 4(3 + \frac{\varepsilon}{4}) \qquad \text{or} \qquad 12 - \varepsilon < 4x < 12 + \varepsilon.$$

That is, $4x$ is in the target of radius ε centered at 12. So, any x in the launch pad of radius $\delta = \varepsilon/4$ centered at $x_0 = 3$ has an output $f(x) = 4x$ in the target of radius ε centered at $L = 12$.

So, for this case we can say that for any target, we have a successful launch pad. We have thus proven the limit statement $\lim_{x\to 3} 4x = 12$.

Note that the key in the previous example was to have a relationship between the target radius ε and the launch pad radius δ that guaranteed the launch pad to be successful for each possible value of ε.

Example 2

Prove that $\lim_{x\to 0} x^2 = 0$.

Solution: In this case, $f(x) = x^2$, $x_0 = 0$, and $L = 0$. Let ε be a target radius. Consider the launch pad with radius $\delta = \sqrt{\varepsilon}$. Suppose x is in this launch pad of radius $\delta = \sqrt{\varepsilon}$ centered at $a = 0$. Then $x \neq 0$ and

$$0 - \sqrt{\varepsilon} < x < 0 + \varepsilon \qquad \text{or} \qquad |x - 0| < \sqrt{\varepsilon}$$

Square both sides to get $|x|^2 < \varepsilon$ or $|x^2| < \varepsilon$. Since $x^2 = x^2 - 0$, we can write the last inequality as $|x^2 - 0| < \varepsilon$. So, $f(x) = x^2$ is in the target of radius ε centered at $L = 0$. We have shown that any x in the launch pad of radius $\delta = \varepsilon$ centered at $x_0 = 0$ has an output $f(x) = x^2$ in the target of radius ε centered at $L = 0$. We have thus proven the limit statement $\lim_{x\to 0} x^2 = 0$.

Version 1 of our definition uses some language that is not standard. To connect with a more common statement of the definition, we need only unpack what we mean by target, launch pad, and successful launch pad. Here's a new version with commentary in square brackets linking to the old version.

Definition (Version 2): The number L is the limit of f at x_0 if for each $\varepsilon > 0$ [that is, for each possible target radius], there is a corresponding number $\delta > 0$ [that is, a launch pad radius] such that

$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |f(x) - L| < \varepsilon$$

[that is, each x in the launch pad has $f(x)$ in the target so the launch pad is successful].

Here's a final version with the commentary removed.

Definition (Version 3): The number L is the limit of f at x_0 if for each $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |f(x) - L| < \varepsilon.$$

This definition (in each of its versions) generalizes easily to functions $f : \mathbb{R}^n \to \mathbb{R}$. In the problems, you are asked to write generalizations for the cases $f : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$.

Problems

1. (a) Generalize the definition of *target* to the case of functions $f : \mathbb{R}^2 \to \mathbb{R}$. Sketch a generic target that includes appropriate labels.

   (b) Generalize the definition of *launch pad* to the case of functions $f : \mathbb{R}^2 \to \mathbb{R}$. Sketch a generic launch pad that includes appropriate labels.

   (c) Generalize the definition of *successful launch pad* to the case of functions $f : \mathbb{R}^2 \to \mathbb{R}$.

   (d) Write a generalization of Version 1 of the definition for functions $f : \mathbb{R}^2 \to \mathbb{R}$.

   (e) Write a generalization of Version 3 of the definition for functions $f : \mathbb{R}^2 \to \mathbb{R}$.

2. (a) Generalize the definition of *target* to the case of functions $f : \mathbb{R}^3 \to \mathbb{R}$. Sketch a generic target that includes appropriate labels.

   (b) Generalize the definition of *launch pad* to the case of functions $f : \mathbb{R}^3 \to \mathbb{R}$. Sketch a generic launch pad that includes appropriate labels.

   (c) Generalize the definition of *successful launch pad* to the case of functions $f : \mathbb{R}^3 \to \mathbb{R}$.

   (d) Write a generalization of Version 1 of the definition for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

   (e) Write a generalization of Version 3 of the definition for functions $f : \mathbb{R}^3 \to \mathbb{R}$.