Fundamental Theorems

Math 280

Spring 2011

Fundamental Theorem for Definite Integrals

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

By substituting, we can also write the conclusion as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Note: In the above and following theorems, a hypothesis on continuity of the integrand is omitted in order to focus on other details.

Fundamental Theorem for Line Integrals

Let C be a curve that starts at A and ends at B. If $\vec{\nabla} V = \vec{F}$, then

$$\int\limits_C \vec{F} \cdot d\vec{r} = V(B) - V(A).$$

By substituting, we can also write the conclusion as

$$\int\limits_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A).$$

The Divergence Theorem

Let D be a solid region with the closed surface S as the edge of D and area element vectors $d\vec{A}$ for S oriented outward. If $\vec{\nabla} \cdot \vec{F} = f$, then

$$\iiint\limits_D f \, dV = \oiint\limits_S \vec{F} \cdot d\vec{A}.$$

By substituting, we can also write the conclusion as

$$\iiint\limits_D (\vec{\nabla} \cdot \vec{F}) \, dV = \oiint\limits_S \vec{F} \cdot d\vec{A}.$$

Stokes' Theorem

Let S be a surface with the closed curve C as the edge of S. Orient the area element vectors $d\vec{A}$ and the curve C to have a right-hand relation. If $\vec{\nabla} \times \vec{F} = \vec{G}$, then

$$\iint_S \vec{G} \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}.$$

By substituting, we can also write the conclusion as

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}.$$

Green's Theorem (as a special case of Stokes' Theorem)

Start with $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} + 0\,\hat{k}$.

$$\vec{\nabla} \times \vec{F} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k}.$$

D: region in the xy-plane with closed curve C as edge.
Orient curve C counterclockwise.
Express area element vectors as $d\vec{A} = dx\,dy\,\hat{k}$.

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\hat{k} \cdot dx\,dy\,\hat{k} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy.$$

Stokes' Theorem for this case:

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dx\,dy = \oint_C \big(P\,\hat{\imath} + Q\,\hat{\jmath}\big) \cdot d\vec{r}.$$

Green's Theorem (alternate notation)

Using an alternate notation for line integrals, Green's Theorem can also be written as

$$\iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx\, dy = \oint\limits_C P\, dx + Q\, dy.$$

All together now

FTC	$\int_a^b F'(x)\,dx = F(b) - F(a)$
FTC for line integrals	$\int\limits_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A)$
Divergence	$\iiint\limits_D (\vec{\nabla} \cdot \vec{F})\,dV = \oiint\limits_S \vec{F} \cdot d\vec{A}$
Stokes'	$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{r}$
Green's	$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint\limits_C P\,dx + Q\,dy.$