Equations of planes

You should be familiar with equations of lines in the plane. From this experience, you know that the equation of a line in the plane is a *linear equation in two variables.* We'll use x and y as the two variables. As an example, consider the equation

$$3x + 4y - 8 = 0.$$

This form of the equation is called the *standard form.* We can algebraically manipulate this into other forms such as the *slope-intercept form*

$$y = -\frac{3}{4}x + 2$$

or a *point-slope form*

$$y - 5 = -\frac{3}{4}(x + 4).$$

[Note that there are many point-slope forms depending on which point we choose to focus attention. Here, the point $(-4, 5)$ was chosen as the focus of attention.] Each of these forms is useful in different contexts. In calculus, a point-slope form is often useful in writing the equation of a tangent line since we most often have information about a point on the tangent (from the function) and the slope of the tangent line (from the derivative of the function).

More generally, we can express these forms as

$Ax + By + C = 0$	standard form
$y = mx + b$	slope-intercept form
$y-y_0 = m(x - x_0)$	point-slope form

You are probably comfortable with reading off geometric information from the latter two equations. We will see later that the constants A and B in the standard form can also be given direct geometric interpretation.

Planes in space are described by *linear equations in three variables.* For example, consider the equation

$$3x + 4y - 2z - 12 = 0.$$

The set of all points with cartesian coordinates (x, y, z) that satisfy this equation form a particular plane. We can read off geometric information about this plane if we solve for z to get

$$z = \frac{3}{2}x + 2y - 6.$$

This is the *slopes-intercept* form for the equation of this plane. Note that *slopes* is plural here since we have *two* slopes. The coefficient $3/2$ is the x-slope and the coefficient 2 is the y-slope. We'll denote these m_x and m_y so here we have

$$m_x = \frac{3}{2} \qquad \text{and} \qquad m_y = 2.$$

The x-slope is a "rise over run" with y held constant and, in similar fashion, the y-slope is "rise over run" with x held constant. To be more detailed, we have

$$m_x = \frac{\text{rise in } z}{\text{run in } x} \quad \text{with } y \text{ held constant}$$

and

$$m_y = \frac{\text{rise in } z}{\text{run in } y} \quad \text{with } x \text{ held constant.}$$

[Note that the rise is a change in z for both of these since we have singled out the z coordinate by solving the original equation for this variable.] So, for this example, we have a rise of 3 units in the z direction for any run of 2 units in the x direction with y kept constant. Similarly, by thinking of 2 as $2/1$, we have a rise of 2 units in the z direction for any run of 1 unit in the y direction with x kept constant.

The two slopes $m_x = 3/2$ and $m_y = 2$ give us the orientation of the plane. The constant term -6 in the equation is the z-intercept (since the equation gives $z = -6$ with $x = 0$ and $y = 0$). The z-intercept picks out one particular plane in the stack of parallel planes having slopes $m_x = 3/2$ and $m_y = 2$.

More generally, we can express the equation of a plane in any one of several forms:

$$\begin{array}{cl} Ax + By + Cz + D = 0 & \text{standard form} \\ z = m_x x + m_y y + b & \text{slopes-intercept form} \\ z - z_0 = m_x(x - x_0) + m_y(y - y_0) & \text{point-slopes form} \end{array}$$

Example

Find the standard form of the equation for the plane that contains the points $P(2, 5, 0)$, $Q(4, 5, 6)$, and $R(2, 3, 4)$.

We start by noting that y is constant between the points P and Q so we can use these two points to compute

$$m_x = \frac{6-0}{4-2} = \frac{6}{2} = 3.$$

In similar fashion, we note that x is constant between points P and R so we can compute

$$m_y = \frac{4-0}{3-5} = \frac{4}{-2} = -2.$$

We can now look at the point-slopes form using these slopes together with any one of the three given points. Choosing P, we get

$$z - 0 = 3(x - 2) - 2(y - 5).$$

With some algebra, we can manipulate this into the standard form

$$3x - 2y - z + 4 = 0.$$

As a check, we can verify that each of the three given points satisfies this equation.

Problems on equations of planes

1. Determine which, if any, of the following points are on the plane having equation $2x - y + 6z = 14$.

   (a) $(5, -4, 0)$ (b) $(1, 6, 2)$ (c) $(2, 8, 3)$

   Answer: $(5, -4, 0)$ and $(2, 8, 3)$ on plane, $(1, 6, 2)$ not

2. Determine the x-intercept, the y-intercept, and the z-intercept of the plane having equation $2x - y + 6z = 14$.

   Answer: $(7, 0, 0)$, $(0, -14, 0)$ and $(0, 0, 7/3)$

3. Determine the slopes of the plane having equation $2x - y + 6z = 14$.

   Answer: $m_x = -1/3$ and $m_y = 1/6$

4. Find the standard form equation for the plane containing the point $(2, -6, 1)$ with slopes $m_x = 3$ and $m_y = -2$.

   Answer: $3x - 2y - z = 17$

5. Find an equation for the plane that contains the points $(0, 0, 0)$, $(2, 0, 6)$, and $(0, 5, 20)$.

   Answer: $z = 3x + 4y$

6. Find an equation for the plane that contains the points $(0, 0, 0)$, $(0, 4, -8)$, and $(3, 0, 6)$.

   Answer: $z = 2x - 2y$

7. Find an equation for the plane that contains the points $(1, 3, 2)$, $(1, 7, 10)$, and $(3, 3, 8)$.

   Answer: $z = 3x + 2y - 7$ or $z - 2 = 3(x - 1) + 2(y - 3)$ or...

8. Find an equation for the plane that contains the points $(7, 2, 1)$, $(5, 2, -4)$, and $(5, -2, 10)$.

9. (*Challenge problem*) Find an equation for the plane that contains the points $(1, 3, 2)$, $(1, 7, 10)$, and $(4, 2, 1)$.

   Answer: $z = \frac{1}{3}x + 2y - \frac{13}{3}$ or $z - 2 = \frac{1}{3}(x - 1) + 2(y - 3)$ or...

10. (*Challenge problem*) Find an equation for the plane that contains the points $(1, 3, 2)$, $(5, 7, 10)$, and $(4, 2, 1)$.

    Answer: $z = \frac{1}{4}x + \frac{7}{4}y - \frac{7}{2}$ or $z - 2 = \frac{1}{4}(x - 1) + \frac{7}{4}(y - 3)$ or...