Derivatives of vector fields: divergence and curl

Vector fields in the plane

Given a planar vector field $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$, we can consider the partial derivatives

$$\begin{array}{cc} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y}. \end{array}$$

Each of these tells us about the rate of change in a component of $\vec{F}$ with respect to displacement in one of the coordinate directions. For example, $\partial P/\partial y$ gives the rate of change in the horizontal component of $\vec{F}$ with respect to displacement in the vertical direction. There are two combinations of these four partial derivatives that are most useful. We will think about these in the context in which we consider $\vec{F}$ to be the velocity at each point in the plane for a flowing fluid.

If we go to a fixed point in the plane and consider an infinitesimal box, we can ask whether the fluid flow is causing fluid to empty out of the box or accumulate in the box. As we discussed in class, the partial derivatives $\partial P/\partial x$ and $\partial Q/\partial y$ evaluated at that fixed point are independent contributions to the rate at which fluid is emptying out of the box. So, the sum $\partial P/\partial x + \partial Q/\partial y$ is a total rate of at which fluid is emptying out of the box. The combination $\partial P/\partial x + \partial Q/\partial y$ is called the *divergence of the vector field* $\vec{F}$. We denote this

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The divergence $\operatorname{div}\vec{F} = \partial P/\partial x + \partial Q/\partial y$ is an *area rate density* (also be referred to as a *flux density*).

Example 1

The divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ is

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[y] = 1 + 1 = 2.$$

So, fluid flow with this radially outward velocity field will empty fluid out of an infinitesimal box at the rate density of 2 everywhere in the plane. To associate units with this, let's use meters per second (m/s) for velocities. The units associated with the value of 2 are then square meters per second per square meter ($(\mathrm{m}^2/\mathrm{s})/\mathrm{m}^2$). Note that there is some cancellation here so we could write the result as

$$2\frac{\mathrm{m}^2/\mathrm{s}}{\mathrm{m}^2} = 2\frac{\mathrm{m}^2/\mathrm{m}^2}{s} = 2\frac{\%}{s} = 2\% \text{ of the fluid area per second.}$$

One way to conceptualize fluid flowing out of the infinitesimal box at a rate of 2% of the fluid area per second is to think of the fluid as becoming less dense at each point.

We now consider another aspect of the fluid flow, namely the effect on an infinitesimal paddlewheel at a fixed point in the plane. The axis of the paddlewheel is in the direction $\hat{k}$. As we discussed in class, the partial derivatives $\partial Q/\partial x$ and $-\partial P/\partial y$ measure independent contributions to the rotation rate of the paddlewheel (with a positive rate corresponding to counterclockwise rotation). So, the combination $\partial Q/\partial x - \partial P/\partial y$ measures the total rotation rate. The combination $\partial Q/\partial x - \partial P/\partial y$ is the $\hat{k}$ component of what we call the *curl of the vector field* $\vec{F}$. We denote this

$$\operatorname{curl} \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$

Example 2

The curl of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ is

$$\operatorname{curl} \vec{F} = \left(\frac{\partial}{\partial y}[x] - \frac{\partial}{\partial x}[y]\right)\hat{k} = (0-0)\,\hat{k} = \vec{0}.$$

Since the curl is zero at every point in the plane, a fluid with this velocity field will not produce rotation in a paddlewheel at any point.

Vector fields in space

For a vector field in space, we have $\vec{F} = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$. We thus have nine partial derivatives to consider:

$$\begin{array}{ccc} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} & \dfrac{\partial P}{\partial z} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} & \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} & \dfrac{\partial R}{\partial z} \end{array}$$

Divergence and curl are particular combinations of these partial derivatives.

The divergence of a vector field in space is not surprising: we simply add a third contribution to the "box emptying out rate" corresponding to the third coordinate direction. So, we have

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

For a vector field in space, we can think of divergence is an *volume rate density* (or a *flux density*.

Example 3

The divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ is

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[y] + \frac{\partial}{\partial z}[z] = 1+1+1 = 3.$$

So, fluid flow with this radially outward velocity field will empty fluid out of an infinitesimal box at the rate density of 3% of the fluid volume everywhere in space.

To generalize curl to a vector field in space, we need to recognize that the axis of the conceptual paddlewheel can point in any direction. For a paddlewheel with axis in the direction $\hat{k}$, the rotation rate is is given by $\partial Q/\partial x - \partial P/\partial y$. In similar fashion, for a paddlewheel with axis in the $\hat{\imath}$ direction has a rotation rate given by $\partial R/\partial y - \partial Q/\partial z$. For a paddlewheel with axis in the $\hat{\jmath}$ direction, the rotation rate is $-\partial R/\partial x + \partial P/\partial z$. Putting this all together, we define the curl of a vector field in space as

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{k}.$$

Example 4

The curl of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ is

$$\begin{aligned} \operatorname{curl} \vec{F} &= \left(\frac{\partial}{\partial y}[z] - \frac{\partial}{\partial z}[y]\right) \hat{\imath} - \left(\frac{\partial}{\partial x}[z] - \frac{\partial}{\partial z}[x]\right) \hat{\jmath} + \left(\frac{\partial}{\partial x}[y] - \frac{\partial}{\partial y}[x]\right) \hat{k} \\ &= 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} = \vec{0}. \end{aligned}$$

So, fluid flow with this radially outward velocity field will not rotate a paddlewheel at any point.

Example 5

The curl of the vector field $\vec{F} = yz\,\hat{\imath} + xz\,\hat{\jmath} - xy\,\hat{k}$ is

$$\begin{aligned} \operatorname{curl} \vec{F} &= \left(\frac{\partial}{\partial y}[-xy] - \frac{\partial}{\partial z}[xz]\right) \hat{\imath} - \left(\frac{\partial}{\partial x}[-xy] - \frac{\partial}{\partial z}[yz]\right) \hat{\jmath} + \left(\frac{\partial}{\partial x}[xz] - \frac{\partial}{\partial y}[yz]\right) \hat{k} \\ &= (-x - x)\,\hat{\imath} - (-y - y)\,\hat{\jmath} + (z - z)\,\hat{k} = -2x\,\hat{\imath} + 2y\,\hat{\jmath} + 0\,\hat{k}. \end{aligned}$$

To get some feel for this, let's evaluate the curl at a specific point, say $(x, y, z) = (1, 2, 3)$. We get

$$\operatorname{curl} \vec{F}(1, 2, 3) = -2\,\hat{\imath} + 4\,\hat{\jmath} + 0\,\hat{k}.$$

So, at the point $(1, 2, 3)$, a paddlewheel with axis oriented in the direction $\hat{\imath}$ will rotate at the rate -2. If we orient the paddlewheel so its axis is in the direction $\hat{\jmath}$, the rotation rate will be 4. With the paddlewheel axis in the direction $\hat{k}$, the fluid flow results in no rotation.

From the curl vector, we can read off directly the rotation rates for a paddlewheel oriented with its axis in any one of the three coordinate directions. What about a more general orientation? Let $\hat{p}$ be a unit vector giving the direction of the paddlewheel axis. The rotation rate is given by the component of the curl in the axis direction $\hat{p}$. We can compute this as a dot product so

$$(\operatorname{curl} \vec{F}) \cdot \hat{p} = \text{ rotation rate of a paddlewheel with axis direction } \hat{p}.$$

Example 6

For the vector field in Example 5, consider a paddlewheel at the point $(1,2,3)$ with axis direction $\hat{p} = (2\hat{\imath} + 3\,\hat{\jmath})/\sqrt{13}$. Using the curl vector we previously computed, we find that the rotation rate for this paddlewheel orientation is

$$\left(\operatorname{curl} \vec{F}(1,2,3)\right) \cdot \hat{p} = (-2\,\hat{\imath} + 4\,\hat{\jmath} + 0\,\hat{k}) \cdot \left(\frac{2\hat{\imath} + 3\hat{\jmath}}{\sqrt{13}}\right) = \frac{-4+12}{\sqrt{13}} = \frac{8}{\sqrt{13}}.$$

Given the idea that

$$(\operatorname{curl} \vec{F}) \cdot \hat{p} = \text{ rotation rate of a paddlewheel with axis direction } \hat{p}$$

note that the greatest rotation rate comes when the paddlewheel axis is in the same direction as the curl vector. So, we have the following interpretation: At each point in the domain of $\vec{F}$, the vector $\operatorname{curl} \vec{F}$ is a vector with

- direction of $\operatorname{curl} \vec{F}$ is the direction to orient a paddlewheel for the greatest rotation rate
- magnitude of $\operatorname{curl} \vec{F}$ is that greatest rotation rate

Note how this parallels our interpretation of a gradient vector $\vec{\nabla} f$ as pointing in the direction of greatest rate of change in f with magnitude equal to that greatest rate.

Example 7

For the vector field in Example 5, a paddlewheel at the point $(1,2,3)$ will rotate fastest when oriented with axis in the direction $\hat{p} = (-2\hat{\imath} + 4\,\hat{\jmath})/\sqrt{20} = (-\hat{\imath} + 2\,\hat{\jmath})/\sqrt{5}$. With this paddlewheel orientation, the rotation rate is

$$\|\operatorname{curl} \vec{F}(1,2,3)\| = \|-2\,\hat{\imath} + 4\,\hat{\jmath} + 0\,\hat{k}\| = \sqrt{20} = 2\sqrt{5}.$$

The $\vec{\nabla}$ operator viewpoint

We define the vector operator $\vec{\nabla}$ as

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

With this, we can express the gradient of a scalar function f as

$$\vec{\nabla} f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$$

The operator $\vec{\nabla}$ can act on a vector field $\vec{F}$ either with a dot product or with a cross product. Using a dot product produces the divergence:

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div} \vec{F}.$$

Using a cross product produces the curl:

$$\vec{\nabla}\times\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(P\,\hat{\imath}+Q\,\hat{\jmath}+R\,\hat{k}) = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\hat{\imath}-\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\hat{\jmath}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{k}.$$

So, the $\vec{\nabla}$ operator point of view provides a unified way of computing gradient, divergence, and curl. This viewpoint makes certain rules easy and natural to state. For example, if we have a scalar function f and a vector field $\vec{F}$, we have a natural product rule for the divergence:

$$\vec{\nabla}\cdot(f\vec{F}) = \vec{\nabla}f\cdot\vec{F} + f\,\vec{\nabla}\cdot\vec{F}.$$

Note that the right side of this product rule has the form "derivative of the first times the second plus first times the derivative of the second".

Problems: divergence and curl

1. For each of the following vector fields, compute the divergence. Evaluate the divergence at a few points and give an interpretation for each value.

   (a) $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ *Answer:* $\vec{\nabla} \cdot \vec{F} = 2$

   (b) $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ *Answer:* $\vec{\nabla} \cdot \vec{F} = 3$

   (c) $\vec{F} = z\sin(xy)\,\hat{\imath} + (x+y)\,\hat{\jmath} + ze^x\,\hat{k}$
   Answer: $\vec{\nabla} \cdot \vec{F} = yz\cos(xy) + 1 + e^x$

   (d) $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$ *Answer:* $\vec{\nabla} \cdot \vec{F} = 0$

   (e) $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath}}{\sqrt{x^2+y^2}}$ *Answer:* $\vec{\nabla} \cdot \vec{F} = 1/\sqrt{x^2+y^2}$

   (f) $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath}}{x^2+y^2}$ *Answer:* $\vec{\nabla} \cdot \vec{F} = 0$

2. For each of the following vector fields, compute the curl. Evaluate the curl at a few points and give an interpretation for each value.

   (a) $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$ *Answer:* $\vec{\nabla} \times \vec{F} = \vec{0}$

   (b) $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ *Answer:* $\vec{\nabla} \times \vec{F} = \vec{0}$

   (c) $\vec{F} = z\sin(xy)\,\hat{\imath} + (x+y)\,\hat{\jmath} + ze^x\,\hat{k}$
   Answer: $\vec{\nabla} \times \vec{F} = \big(ze^x + \sin(xy)\big)\,\hat{\jmath} + \big(1 + xz\cos(xy)\big)\,\hat{k}$

   (d) $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$ *Answer:* $\vec{\nabla} \times \vec{F} = 2\,\hat{k}$

   (e) $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\sqrt{x^2+y^2}}$ *Answer:* $\vec{\nabla} \times \vec{F} = (1/\sqrt{x^2+y^2})\,\hat{k}$

   (f) $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2}$ *Answer:* $\vec{\nabla} \times \vec{F} = \vec{0}$

3. Let $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ be a vector field with the component functions P, Q, and R having continuous second partial derivatives. Show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.
   Answer: See next page.

4. Let f be a function with continuous second partial derivatives. Show that $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$.
   Answer: See next page.

3. Let $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ be a vector field with the component functions P, Q, and R having continuous second partial derivatives. Show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

Solution:

$$
\begin{aligned}
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 P}{\partial z \partial y}\right) \\
&= 0 \qquad \text{by equality of mixed second partial derivatives}
\end{aligned}
$$

4. Let f be a function with continuous second partial derivatives. Show that $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$.

Solution:

$$
\begin{aligned}
\vec{\nabla} \times \vec{\nabla} f &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}\right) \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{\imath} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\hat{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{k} \\
&= 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} \qquad \text{by equality of mixed second partial derivatives} \\
&= \vec{0}
\end{aligned}
$$