

Conservative vector fields

Let $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ be a vector field that is continuous on some domain in \mathbb{R}^3 . Let D be an open, connected, and simply connected region in the domain of \vec{F} . The following conditions are equivalent (that is, any one of the conditions implies any of the other conditions):

1. There is a function V such that $\vec{\nabla}V = \vec{F}$ for all point in D .
2. The following equations hold for all points in D :

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

These equations are equivalent to the condition

$$\vec{\nabla} \times \vec{F} = \vec{0}.$$

3. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for every pair of smooth curves in C_1 and C_2 in D that have a common start point and a common end point.
4. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every smooth closed curve C in D

A vector field satisfying any one (and hence all) of these properties is said to be *conservative* for the region D . A function V satisfying the first property with respect to \vec{F} is called a *potential function* for the vector field \vec{F} . The second property is called the *component test*. The third property is referred to as *path-independance*. The fourth property is sometimes called the *closed-loop* property.

The four properties are also equivalent for a vector field $\vec{F} = P\hat{i} + Q\hat{j}$ with domain in \mathbb{R}^2 . In this case, the component test is simply

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$