3. Use definition (1), Sec. 32, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing
   (*a*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;
   (*b*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal nth root* of a nonzero complex number z_0, defined in Sec. 8, is the same as the principal value of $z_0^{1/n}$, defined in Sec. 32.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a \ln |z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

   Ans. c is real.

8. Let c, d, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then
   (*a*) $1/z^c = z^{-c}$; (*b*) $(z^c)^n = z^{cn}$ $(n = 1, 2, \ldots)$;
   (*c*) $z^c z^d = z^{c+d}$; (*d*) $z^c/z^d = z^{c-d}$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

33. TRIGONOMETRIC FUNCTIONS

Euler's formula (Sec. 6) tells us that

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

for every real number x, and it follows from these equations that

$$e^{ix} - e^{-ix} = 2i \sin x \quad \text{and} \quad e^{ix} + e^{-ix} = 2 \cos x.$$

That is,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

It is, therefore, natural to *define* the sine and cosine functions of a complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{1}$$

These functions are entire since they are linear combinations (Exercise 3, Sec. 24) of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives of those exponential functions, we find from equations (1) that

$$\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z. \tag{2}$$

It is easy to see from definitions (1) that

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z; \tag{3}$$

and a variety of other identities from trigonometry are valid with complex variables.

EXAMPLE. In order to show that

$$2 \sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2), \tag{4}$$

using definitions (1) and properties of the exponential function, we first write

$$2 \sin z_1 \cos z_2 = 2 \left(\frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left(\frac{e^{iz_2} - e^{-iz_2}}{2} \right).$$

Multiplication then reduces the right-hand side here to

$$\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} + \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i},$$

or

$$\sin(z_1 + z_2) + \sin(z_1 - z_2);$$

and identity (4) is established.

Identity (4) leads to the identities (see Exercises 3 and 4)

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{5}$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2; \tag{6}$$

and from these it follows that

$$\sin^2 z + \cos^2 z = 1, \tag{7}$$

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \tag{8}$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z. \tag{9}$$

When y is any real number, one can use definitions (1) and the hyperbolic functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

from calculus to write

$$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y. \tag{10}$$

The real and imaginary parts of $\sin z$ and $\cos z$ are then readily displayed by writing $z_1 = x$ and $z_2 = iy$ in identities (5) and (6):

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{11}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \tag{12}$$

where $z = x + iy$.

A number of important properties of $\sin z$ and $\cos z$ follow immediately from expressions (11) and (12). The periodic character of these functions, for example, is evident:

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z, \tag{13}$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z. \tag{14}$$

Also (see Exercise 9)

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{15}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{16}$$

Inasmuch as $\sinh y$ tends to infinity as y tends to infinity, it is clear from these two equations that $\sin z$ and $\cos z$ are *not bounded* on the complex plane, whereas the absolute values of $\sin x$ and $\cos x$ are less than or equal to unity for all values of x. (See the definition of boundedness at the end of Sec. 17.)

A *zero* of a given function $f(z)$ is a number z_0 such that $f(z_0) = 0$. Since $\sin z$ becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are all zeros of $\sin z$. To show that *there are no other zeros,* we assume that $\sin z = 0$ and note how it follows from equation (15) that

$$\sin^2 x + \sinh^2 y = 0.$$

Thus

$$\sin x = 0 \quad \text{and} \quad \sinh y = 0.$$

Evidently, then, $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $y = 0$; that is,

$$\sin z = 0 \quad \text{if and only if} \quad z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots). \tag{17}$$

Since

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

according to the second of identities (9),

$$\cos z = 0 \quad \text{if and only if} \quad z = \frac{\pi}{2} + n\pi \ (n = 0, \pm 1, \pm 2, \ldots). \tag{18}$$

So, as was the case with $\sin z$, the zeros of $\cos z$ are all real.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the usual relations:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \tag{19}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}. \tag{20}$$

Observe that the quotients $\tan z$ and $\sec z$ are analytic everywhere except at the singularities (Sec. 23)

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which are the zeros of $\cos z$. Likewise, $\cot z$ and $\csc z$ have singularities at the zeros of $\sin z$, namely

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (19) and (20), we obtain the expected differentiation formulas

$$\frac{d}{dz} \tan z = \sec^2 z, \qquad \frac{d}{dz} \cot z = -\csc^2 z, \tag{21}$$

$$\frac{d}{dz} \sec z = \sec z \tan z, \qquad \frac{d}{dz} \csc z = -\csc z \cot z. \tag{22}$$

The periodicity of each of the trigonometric functions defined by equations (19) and (20) follows readily from equations (13) and (14). For example,

$$\tan(z + \pi) = \tan z. \tag{23}$$

Mapping properties of the transformation $w = \sin z$ are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Sec. 89 (Chap. 8), where they are discussed.

EXERCISES

1. Give details in the derivation of expressions (2), Sec. 33, for the derivatives of $\sin z$ and $\cos z$.
2. Show that Euler's formula (Sec. 6) continues to hold when θ is replaced by z:

$$e^{iz} = \cos z + i \sin z.$$

Suggestion: To verify this, start with the right-hand side.

3. In Sec. 33, interchange z_1 and z_2 in equation (4) and then add corresponding sides of the resulting equation and equation (4) to derive expression (5) for $\sin(z_1 + z_2)$.

4. According to equation (5) in Sec. 33,

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive expression (6) for $\cos(z_1 + z_2)$ in that section.

5. Verify identity (7) in Sec. 33 using

(*a*) identity (6) and relations (3) in that section;

(*b*) the lemma in Sec. 26 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

6. Show how each of the trigonometric identities (8) and (9) in Sec. 33 follows from one of the identities (5) and (6) in that section.

7. Use identity (7) in Sec. 33 to show that

(*a*) $1 + \tan^2 z = \sec^2 z$; (*b*) $1 + \cot^2 z = \csc^2 z$.

8. Establish differentiation formulas (21) and (22) in Sec. 33.

9. In Sec. 33, use expressions (11) and (12) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

10. Point out how it follows from expressions (15) and (16) in Sec. 33 for $|\sin z|^2$ and $|\cos z|^2$ that

(*a*) $|\sin z| \geq |\sin x|$; (*b*) $|\cos z| \geq |\cos x|$.

11. With the aid of expressions (15) and (16) in Sec. 33 for $|\sin z|^2$ and $|\cos z|^2$, show that

(*a*) $|\sinh y| \leq |\sin z| \leq \cosh y$; (*b*) $|\sinh y| \leq |\cos z| \leq \cosh y$.

12. (*a*) Use definitions (1), Sec. 33, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(*b*) With the aid of the identity obtained in part (*a*), show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

13. Use the Cauchy–Riemann equations and the theorem in Sec. 20 to show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

14. Use the reflection principle (Sec. 27) to show that, for all z,

(*a*) $\overline{\sin z} = \sin \bar{z}$; (*b*) $\overline{\cos z} = \cos \bar{z}$.

15. With the aid of expressions (11) and (12) in Sec. 33, give direct verifications of the relations obtained in Exercise 14.

16. Show that

(*a*) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(*b*) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

17. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\frac{\pi}{2} + 2n\pi\right) \pm 4i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

18. Find all roots of the equation $\cos z = 2$.

Ans. $2n\pi + i \cosh^{-1} 2$, or $2n\pi \pm i \ln(2 + \sqrt{3})$ $(n = 0, \pm 1, \pm 2, \ldots)$.

34. HYPERBOLIC FUNCTIONS

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$\frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z. \tag{2}$$

Because of the way in which the exponential function appears in definitions (1) and in the definitions (Sec. 33)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

of $\sin z$ and $\cos z$, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \tag{3}$$

$$-i \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z. \tag{4}$$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \tag{5}$$

$$\cosh^2 z - \sinh^2 z = 1, \tag{6}$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \tag{7}$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{8}$$