CHAPTER

3

ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

28. THE EXPONENTIAL FUNCTION

As anticipated earlier (Sec. 13), we define here the exponential function e^z by writing

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

where Euler's formula (see Sec. 6)

$$e^{iy} = \cos y + i \sin y \tag{2}$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$; and, following the convention used in calculus, we often write $\exp z$ for e^z.

Note that since the *positive* nth root $\sqrt[n]{e}$ of e is assigned to e^x when $x = 1/n$ $(n = 2, 3, \ldots)$, expression (1) tells us that the complex exponential function e^z is also $\sqrt[n]{e}$ when $z = 1/n$ $(n = 2, 3, \ldots)$. This is an exception to the convention (Sec. 8) that would ordinarily require us to interpret $e^{1/n}$ as the set of nth roots of e.

According to definition (1), $e^x e^{iy} = e^{x+iy}$; and, as already pointed out in Sec. 13, the definition is suggested by the additive property

$$e^{x_1}e^{x_2} = e^{x_1+x_2}$$

of e^x in calculus. That property's extension,

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \tag{3}$$

to complex analysis is easy to prove. To do this, we write

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Then

$$e^{z_1}e^{z_2} = (e^{x_1}e^{iy_1})(e^{x_2}e^{iy_2}) = (e^{x_1}e^{x_2})(e^{iy_1}e^{iy_2}).$$

But x_1 and x_2 are both real, and we know from Sec. 7 that

$$e^{iy_1}e^{iy_2} = e^{i(y_1+y_2)}.$$

Hence

$$e^{z_1}e^{z_2} = e^{(x_1+x_2)}e^{i(y_1+y_2)};$$

and, since

$$(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2,$$

the right-hand side of this last equation becomes $e^{z_1+z_2}$. Property (3) is now established

Observe how property (3) enables us to write $e^{z_1-z_2}e^{z_2} = e^{z_1}$, or

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}. \tag{4}$$

From this and the fact that $e^0 = 1$, it follows that $1/e^z = e^{-z}$.

There are a number of other important properties of e^z that are expected. Accord ing to Example 1 in Sec. 21, for instance,

$$\frac{d}{dz}e^z = e^z \tag{5}$$

everywhere in the z plane. Note that the differentiability of e^z for all z tells us th e^z *is entire* (Sec. 23). It is also true that

$$e^z \neq 0 \quad \text{for any complex number } z. \tag{6}$$

This is evident upon writing definition (1) in the form

$$e^z = \rho e^{i\phi} \quad \text{where} \quad \rho = e^x \text{ and } \phi = y,$$

which tells us that

(7) $$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \ (n = 0, \pm 1, \pm 2, \ldots).$$

Statement (6) then follows from the observation that $|e^z|$ is always positive.

Some properties of e^z are, however, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic,* with a pure imaginary period $2\pi i$:

(8) $$e^{z+2\pi i} = e^z.$$

The following example illustrates another property of e^z that e^x does not have. Namely, while e^x is never negative, there are values of e^z that are.

EXAMPLE. There are values of z, for instance, such that

(9) $$e^z = -1.$$

To find them, we write equation (9) as $e^x e^{iy} = 1e^{i\pi}$. Then, in view of the statement in italics at the beginning of Sec. 8 regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = 1 \quad \text{and} \quad y = \pi + 2n\pi \ (n = 0, \pm 1, \pm 2, \ldots).$$

Thus $x = 0$, and we find that

(10) $$z = (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

EXERCISES

1. Show that

   (*a*) $\exp(2 \pm 3\pi i) = -e^2$; (*b*) $\exp\left(\dfrac{2 + \pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1 + i)$;

   (*c*) $\exp(z + \pi i) = -\exp z$.

2. State why the function $2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 20 to show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

4. Show in two ways that the function $\exp(z^2)$ is entire. What is its derivative?

   Ans. $2z \exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

   $$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(*a*) $e^z = -2$; (*b*) $e^z = 1 + \sqrt{3}i$; (*c*) $\exp(2z - 1) = 1$.

Ans. (*a*) $z = \ln 2 + (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $z = \ln 2 + \left(2n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*c*) $z = \dfrac{1}{2} + n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (Compare Exercise 4, Sec. 27.)

10. (*a*) Show that if e^z is real, then $\operatorname{Im} z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$.

(*b*) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as (*a*) x tends to $-\infty$; (*b*) y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \qquad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

14. Establish the identity

$$(e^z)^n = e^{nz} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(*a*) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.

(*b*) Verify it for negative integers n by first recalling from Sec. 7 that

$$z^n = (z^{-1})^m \qquad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part (*a*), together with the property $1/e^z = e^{-z}$ (Sec. 28) of the exponential function.

29. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\Theta} (-\pi < \Theta \leq \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$