CHAPTER

2

ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

11. FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the *domain of definition* of f.*

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

EXAMPLE 1. If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

* Although the domain of definition is often a domain as defined in Sec. 10, it need not be.

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta}),$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{2}$$

EXAMPLE 2. If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently,

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

If, in either of equations (1) and (2), the function v always has value zero, then the value of f is always real. That is, f is a *real-valued function* of a complex variable.

EXAMPLE 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

is a *polynomial* of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of real variables. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

EXAMPLE 4. Let z denote any nonzero complex number. We know from Sec. 8 that $z^{1/2}$ has the two values

$$z^{1/2} = \pm\sqrt{r}\exp\left(i\frac{\Theta}{2}\right),$$

where $r = |z|$ and $\Theta(-\pi < \Theta \le \pi)$ is the *principal value* of $\arg z$. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}\exp\left(i\frac{\Theta}{2}\right) \qquad (r > 0, -\pi < \Theta \le \pi), \tag{3}$$

the (single-valued) function (3) is well defined on the set of nonzero numbers in the z plane. Since zero is the only square root of zero, we also write $f(0) = 0$. The function f is then well defined on the entire plane.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

   (*a*) $f(z) = \dfrac{1}{z^2+1}$; (*b*) $f(z) = \text{Arg}\left(\dfrac{1}{z}\right)$;

   (*c*) $f(z) = \dfrac{z}{z+\bar{z}}$; (*d*) $f(z) = \dfrac{1}{1-|z|^2}$.

   Ans. (*a*) $z \neq \pm i$; (*c*) $\text{Re}\, z \neq 0$.

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

   Ans. $(x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

   to write $f(z)$ in terms of z, and simplify the result.

   Ans. $\bar{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

$$Ans.\ \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

12. MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation, rotation,* and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, the mapping

$$w = iz = r\exp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \overline{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following examples, we illustrate this with the transformation $w = z^2$.

We begin by finding the images of some *curves* in the z plane.

EXAMPLE 1. According to Example 2 in Sec. 11, the mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2, \quad v = 2xy \tag{1}$$

from the xy plane to the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

$$x^2 - y^2 = c_1 \qquad (c_1 > 0) \tag{2}$$

is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations (1) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u = c_1, \quad v = 2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty);$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 17). Likewise, since the pair of equations

$$u = c_1, \quad v = -2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty)$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.

On the other hand, each branch of a hyperbola

$$2xy = c_2 \qquad (c_2 > 0) \tag{3}$$

is transformed into the line $v = c_2$, as indicated in Fig. 17. To verify this, we note from the second of equations (1) that $v = c_2$ when (x, y) is a point on either branch. Suppose

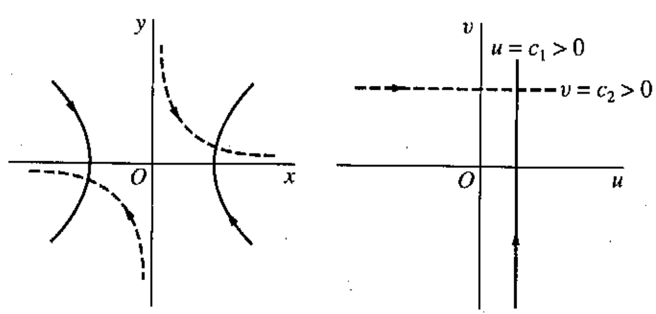

FIGURE 17
$w = z^2$.

that it lies on the branch lying in the first quadrant. Then, since $y = c_2/(2x)$, the first of equations (1) reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2 \qquad (0 < x < \infty).$$

Observe that

$$\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty \quad \text{and} \quad \lim_{x \to \infty} u = \infty.$$

Since u depends continuously on x, then, it is clear that as (x, y) travels down the entire upper branch of hyperbola (3), its image moves to the right along the entire horizontal line $v = c_2$. Inasmuch as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4y^2} - y^2, \quad v = c_2 \qquad (-\infty < y < 0)$$

and since

$$\lim_{y \to -\infty} u = -\infty \quad \text{and} \quad \lim_{\substack{y \to 0 \\ y < 0}} u = \infty,$$

it follows that the image of a point moving *upward* along the entire lower branch also travels to the right along the entire line $v = c_2$ (see Fig. 17).

We shall now use Example 1 to find the image of a certain *region*.

EXAMPLE 2. The domain $x > 0$, $y > 0$, $xy < 1$ consists of all points lying on the upper branches of hyperbolas from the family $2xy = c$, where $0 < c < 2$ (Fig. 18). We know from Example 1 that as a point travels downward along the entirety of one of these branches, its image under the transformation $w = z^2$ moves to the right along the entire line $v = c$. Since, for all values of c between 0 and 2, the branches fill out

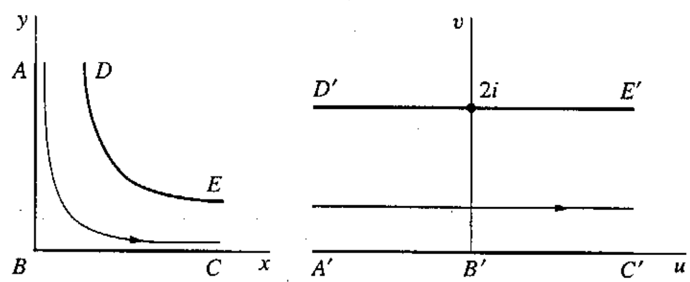

FIGURE 18
$w = z^2$.

the domain $x > 0$, $y > 0$, $xy < 1$, that domain is mapped onto the horizontal strip $0 < v < 2$.

In view of equations (1), the image of a point $(0, y)$ in the z plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \ge 0$, $y \ge 0$, $xy \le 1$ is mapped onto the closed strip $0 \le v \le 2$, as indicated in Fig. 18.

Our last example here illustrates how polar coordinates can be useful in analyzing certain mappings.

EXAMPLE 3. The mapping $w = z^2$ becomes

$$w = r^2 e^{i2\theta}$$

when $z = re^{i\theta}$. Hence if $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = r^2 e^{i2\theta}$; and the statement in italics near the beginning of Sec. 8 tells us that

$$\rho = r^2 \quad \text{and} \quad \phi = 2\theta + 2k\pi,$$

where k has one of the values $k = 0, \pm 1, \pm 2, \ldots$. Evidently, then, the image of any nonzero point z is found by squaring the modulus of z and doubling a value of $\arg z$.

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 19). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $r \ge 0$, $0 \le \theta \le \pi/2$ in the z plane onto the upper half $\rho \ge 0$, $0 \le \phi \le \pi$ of the w plane, as indicated in Fig. 19. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

The transformation $w = z^2$ also maps the upper half plane $r \ge 0$, $0 \le \theta \le \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since

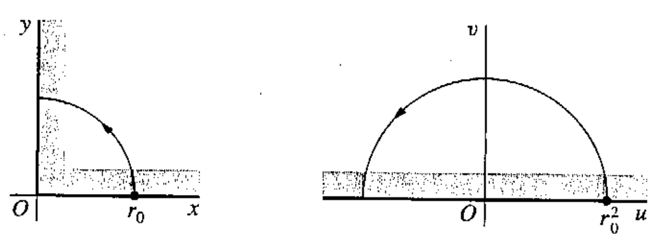

FIGURE 19
$w = z^2$.

both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $\rho e^{i\phi} = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0$, $0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

13. MAPPINGS BY THE EXPONENTIAL FUNCTION

In Chap. 3 we shall introduce and develop properties of a number of elementary functions which do not involve polynomials. That chapter will start with the exponential function

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

the two factors e^x and e^{iy} being well defined at this time (see Sec. 6). Note that definition (1), which can also be written

$$e^{x+iy} = e^x e^{iy},$$

is suggested by the familiar property

$$e^{x_1+x_2} = e^{x_1} e^{x_2}$$

of the exponential function in calculus.

The object of this section is to use the function e^z to provide the reader with additional examples of mappings that continue to be reasonably simple. We begin by examining the images of vertical and horizontal lines.

EXAMPLE 1. The transformation

$$w = e^z \tag{2}$$

can be written $\rho e^{i\phi} = e^x e^{iy}$, where $z = x + iy$ and $w = \rho e^{i\phi}$. Thus $\rho = e^x$ and $\phi = y + 2n\pi$, where n is some integer (see Sec. 8); and transformation (2) can be expressed in the form

$$\rho = e^x, \quad \phi = y. \tag{3}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w plane. That image moves counterclockwise around the circle shown in Fig. 20 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

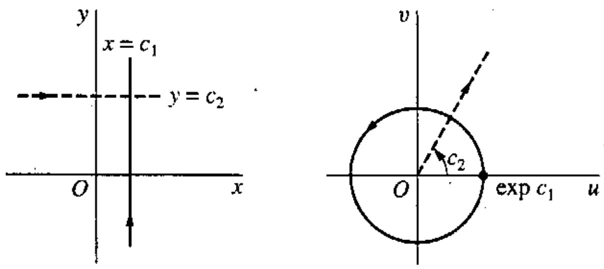

FIGURE 20
$w = \exp z$.

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Evidently, then, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 20.

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

EXAMPLE 2. Let us show that the transformation $w = e^z$ maps the rectangular region $a \le x \le b$, $c \le y \le d$ onto the region $e^a \le \rho \le e^b$, $c \le \phi \le d$. The two regions and corresponding parts of their boundaries are indicated in Fig. 21. The vertical line segment AD is mapped onto the arc $\rho = e^a$, $c \le \phi \le d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b$, $c \le \phi \le d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \le \phi \le \pi$; and the rectangular region is mapped onto half of a circular ring, as shown in Fig. 8, Appendix 2.

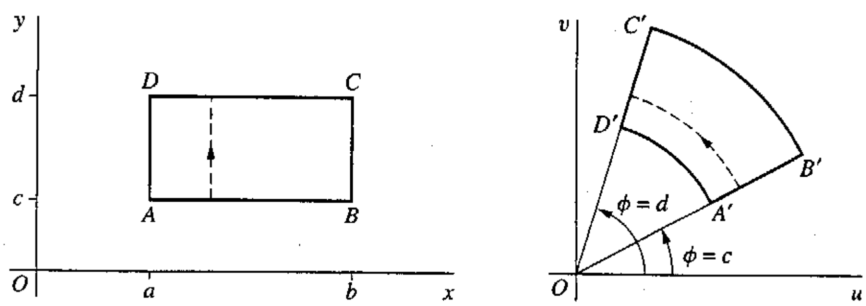

FIGURE 21
$w = \exp z$.

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w plane (Fig. 22). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

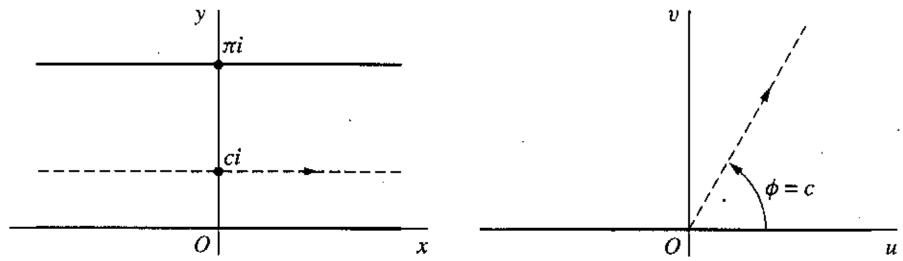

FIGURE 22
$w = \exp z$.

EXERCISES

1. By referring to Example 1 in Sec. 12, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)
2. Find and sketch, showing corresponding orientations, the images of the hyperbolas
$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$
under the transformation $w = z^2$.
3. Sketch the region onto which the sector $r \le 1$, $0 \le \theta \le \pi/4$ is mapped by the transformation (*a*) $w = z^2$; (*b*) $w = z^3$; (*c*) $w = z^4$.
4. Show that the lines $ay = x$ $(a \ne 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.
5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \le x \le b$, $c \le y \le d$ under the transformation $w = \exp z$ is the region $e^a \le \rho \le e^b$, $c \le \phi \le d$, as shown in Fig. 21 (Sec. 13).
6. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.
7. Find the image of the semi-infinite strip $x \ge 0$, $0 \le y \le \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.

8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by (*a*) $w = iz$; (*b*) $w = z/|z|$.

14. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 10) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that, for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that, for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

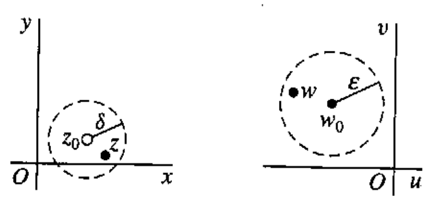

FIGURE 23

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique.* To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for any positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1.$$

So if $0 < |z - z_0| < \delta$, where δ denotes the smaller of the two numbers δ_0 and δ_1, we find that

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \le |f(z) - w_0| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

But $|w_1 - w_0|$ is a nonnegative constant, and ε can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0.$$

Definition (2) requires that f be defined at all points in some deleted neighborhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities (2) need be satisfied by only those points z that lie in both the region *and* the deleted neighborhood.

EXAMPLE 1. Let us show that if $f(z) = iz/2$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}, \tag{3}$$

the point 1 being on the boundary of the domain of definition of f. Observe that when z is in the region $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2}.$$

Hence, for any such z and any positive number ε (see Fig. 24),

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 1| < 2\varepsilon.$$

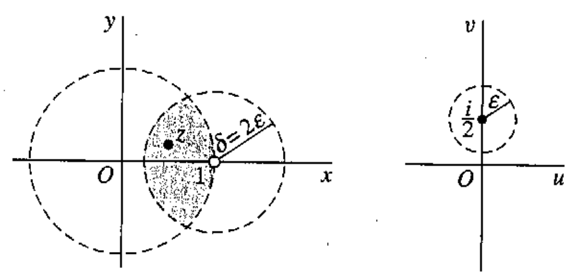

FIGURE 24

Thus condition (2) is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.

If z_0 is an interior point of the domain of definition of f, and limit (1) is to exist, the first of inequalities (2) must hold for *all* points in the deleted neighborhood $0 < |z - z_0| < \delta$. Thus the symbol $z \to z_0$ implies that z is allowed to approach z_0 in an arbitrary manner, not just from some particular direction. The next example emphasizes this.

EXAMPLE 2. If

$$f(z) = \frac{z}{\bar{z}}, \tag{4}$$

the limit

$$\lim_{z \to 0} f(z) \tag{5}$$

does not exist. For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis (Fig. 25),

$$f(z) = \frac{x + i0}{x - i0} = 1;$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = -1.$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit (5) does not exist.

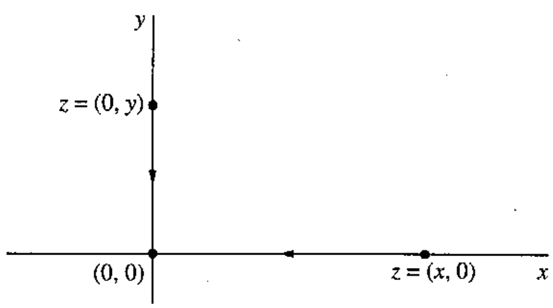

FIGURE 25

While definition (2) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.

15. THEOREMS ON LIMITS

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we use their definition and properties freely.

Theorem 1. *Suppose that*

$$f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 + iy_0, \quad \textit{and} \quad w_0 = u_0 + iv_0.$$

Then

$$\lim_{z \to z_0} f(z) = w_0 \tag{1}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \tag{2}$$

To prove the theorem, we first assume that limits (2) hold and obtain limit (1). Limits (2) tell us that, for each positive number ε, there exist positive numbers δ_1 and δ_2 such that

$$|u - u_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \tag{3}$$

and

$$|v - v_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2. \tag{4}$$

Let δ denote the smaller of the two numbers δ_1 and δ_2. Since

$$|(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| \le |u - u_0| + |v - v_0|$$

and

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - (x_0 + iy_0)|,$$

it follows from statements (3) and (4) that

$$|(u + iv) - (u_0 + iv_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta.$$

That is, limit (1) holds.

Let us now start with the assumption that limit (1) holds. With that assumption, we know that, for each positive number ε, there is a positive number δ such that

$$|(u + iv) - (u_0 + iv_0)| < \varepsilon \tag{5}$$

whenever

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta. \tag{6}$$

But

$$|u - u_0| \le |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|,$$
$$|v - v_0| \le |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|,$$

and

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Hence it follows from inequalities (5) and (6) that

$$|u - u_0| < \varepsilon \quad \text{and} \quad |v - v_0| < \varepsilon$$

whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This establishes limits (2), and the proof of the theorem is complete.

Theorem 2. *Suppose that*

$$\lim_{z \to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z \to z_0} F(z) = W_0. \tag{7}$$

Then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0, \tag{8}$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0; \tag{9}$$

and, if $W_0 \neq 0$,

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}. \tag{10}$$

This important theorem can be proved directly by using the definition of the limit of a function of a complex variable. But, with the aid of Theorem 1, it follows almost immediately from theorems on limits of real-valued functions of two real variables.

To verify property (9), for example, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y),$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0, \quad W_0 = U_0 + iV_0.$$

Then, according to hypotheses (7) and Theorem 1, the limits as (x, y) approaches (x_0, y_0) of the functions u, v, U, and V exist and have the values u_0, v_0, U_0, and V_0, respectively. So the real and imaginary components of the product

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

have the limits $u_0U_0 - v_0V_0$ and $v_0U_0 + u_0V_0$, respectively, as (x, y) approaches (x_0, y_0). Hence, by Theorem 1 again, $f(z)F(z)$ has the limit

$$(u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0)$$

as z approaches z_0; and this is equal to w_0W_0. Property (9) is thus established. Corresponding verifications of properties (8) and (10) can be given.

It is easy to see from definition (2), Sec.14, of limit that

$$\lim_{z\to z_0} c = c \quad \text{and} \quad \lim_{z\to z_0} z = z_0,$$

where z_0 and c are any complex numbers; and, by property (9) and mathematical induction, it follows that

$$\lim_{z\to z_0} z^n = z_0^n \qquad (n = 1, 2, \ldots).$$

So, in view of properties (8) and (9), the limit of a polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

as z approaches a point z_0 is the value of the polynomial at that point:

$$\lim_{z\to z_0} P(z) = P(z_0). \tag{11}$$

16. LIMITS INVOLVING THE POINT AT INFINITY

It is sometimes convenient to include with the complex plane the *point at infinity*, denoted by ∞, and to use limits involving it. The complex plane together with this point is called the *extended* complex plane. To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere centered at the point $z = 0$ (Fig. 26). To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is determined by the intersection of the line through the point z and the north pole N of the sphere with that surface. In

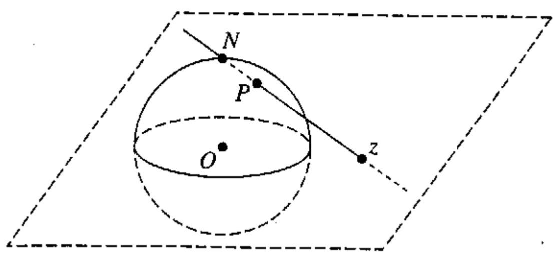

FIGURE 26

like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere,* and the correspondence is called a *stereographic projection.*

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ correspond to points on the sphere close to N. We thus call the set $|z| > 1/\varepsilon$ an ε *neighborhood,* or neighborhood, of ∞.

Let us agree that, in referring to a point z, we mean a point in the *finite* plane. Hereafter, when the point at infinity is to be considered, it will be specifically mentioned.

A meaning is now readily given to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when either z_0 or w_0, or possibly each of these numbers, is replaced by the point at infinity. In the definition of limit in Sec. 14, we simply replace the appropriate neighborhoods of z_0 and w_0 by neighborhoods of ∞. The proof of the following theorem illustrates how this is done.

Theorem. *If z_0 and w_0 are points in the z and w planes, respectively, then*

(1)
$$\lim_{z \to z_0} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

and

(2)
$$\lim_{z \to \infty} f(z) = w_0 \quad \textit{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover,

(3)
$$\lim_{z \to \infty} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z \to 0} \frac{1}{f(1/z)} = 0.$$

We start the proof by noting that the first of limits (1) means that, for each positive number ε, there is a positive number δ such that

(4) $$|f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

That is, the point $w = f(z)$ lies in the ε neighborhood $|w| > 1/\varepsilon$ of ∞ whenever z lies in the deleted neighborhood $0 < |z - z_0| < \delta$ of z_0. Since statement (4) can be written

$$\left|\frac{1}{f(z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta,$$

the second of limits (1) follows.

The first of limits (2) means that, for each positive number ε, a positive number δ exists such that

(5) $$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

Replacing z by $1/z$ in statement (5) and then writing the result as

$$\left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta,$$

we arrive at the second of limits (2).

Finally, the first of limits (3) is to be interpreted as saying that, for each positive number ε, there is a positive number δ such that

(6) $$|f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

When z is replaced by $1/z$, this statement can be put in the form

$$\left|\frac{1}{f(1/z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta;$$

and this gives us the second of limits (3).

EXAMPLES. Observe that

$$\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty \quad \text{since} \quad \lim_{z \to -1} \frac{z + 1}{iz + 3} = 0$$

and

$$\lim_{z \to \infty} \frac{2z + i}{z + 1} = 2 \quad \text{since} \quad \lim_{z \to 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \to 0} \frac{2 + iz}{1 + z} = 2.$$

Furthermore,

$$\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty \quad \text{since} \quad \lim_{z\to 0} \frac{(1/z^2)+1}{(2/z^3)-1} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0.$$

17. CONTINUITY

A function f is *continuous* at a point z_0 if all three of the following conditions are satisfied:

(1) $$\lim_{z\to z_0} f(z) \text{ exists},$$

(2) $$f(z_0) \text{ exists},$$

(3) $$\lim_{z\to z_0} f(z) = f(z_0).$$

Observe that statement (3) actually contains statements (1) and (2), since the existence of the quantity on each side of the equation there is implicit. Statement (3) says that, for each positive number ε, there is a positive number δ such that

(4) $$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

If two functions are continuous at a point, their sum and product are also continuous at that point; their quotient is continuous at any such point where the denominator is not zero. These observations are direct consequences of Theorem 2, Sec. 15. Note, too, that a polynomial is continuous in the entire plane because of limit (11), Sec. 15.

We turn now to two expected properties of continuous functions whose verifications are not so immediate. Our proofs depend on definition (4), and we present the results as theorems.

Theorem 1. *A composition of continuous functions is itself continuous.*

A precise statement of this theorem is contained in the proof to follow. We let $w = f(z)$ be a function that is defined for all z in a neighborhood $|z - z_0| < \delta$ of a point z_0, and we let $W = g(w)$ be a function whose domain of definition contains the image (Sec. 12) of that neighborhood under f. The composition $W = g[f(z)]$ is, then, defined for all z in the neighborhood $|z - z_0| < \delta$. Suppose now that f is continuous at z_0 and that g is continuous at the point $f(z_0)$ in the w plane. In view of the continuity of g at $f(z_0)$, there is, for each positive number ε, a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \varepsilon \quad \text{whenever} \quad |f(z) - f(z_0)| < \gamma.$$

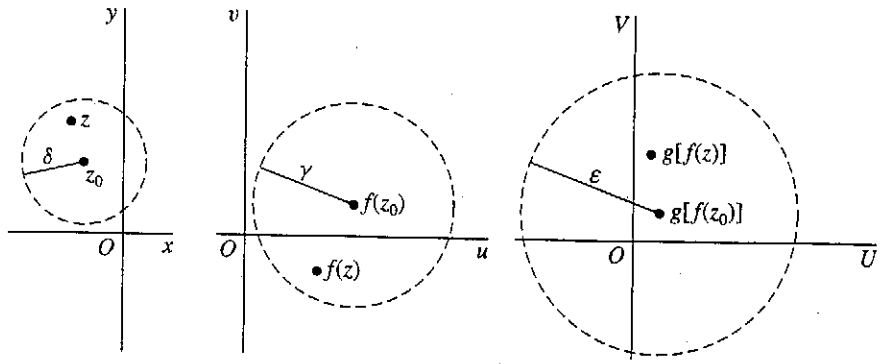

FIGURE 27

(See Fig. 27.) But the continuity of f at z_0 ensures that the neighborhood $|z - z_0| < \delta$ can be made small enough that the second of these inequalities holds. The continuity of the composition $g[f(z)]$ is, therefore, established.

Theorem 2. *If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.*

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)|/2$ to the number ε in statement (4). This tells us that there is a positive number δ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta.$$

So if there is a point z in the neighborhood $|z - z_0| < \delta$ at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2};$$

and the theorem is proved.

The continuity of a function

$$f(z) = u(x, y) + iv(x, y) \tag{5}$$

is closely related to the continuity of its component functions $u(x, y)$ and $v(x, y)$. We note, for instance, how it follows from Theorem 1 in Sec. 15 that *the function (5) is continuous at a point $z_0 = (x_0, y_0)$ if and only if its component functions are* *continuous there.* To illustrate the use of this statement, suppose that the function (5) is continuous in a region R that is both closed and bounded (see Sec. 10). The function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is then continuous in R and thus reaches a maximum value somewhere in that region.* That is, f is *bounded* on R and $|f(z)|$ reaches a maximum value somewhere in R. More precisely, there exists a nonnegative real number M such that

$$|f(z)| \leq M \quad \text{for all } z \text{ in } R, \tag{6}$$

where equality holds for at least one such z.

EXERCISES

1. Use definition (2), Sec. 14, of limit to prove that

(*a*) $\lim_{z \to z_0} \text{Re}\, z = \text{Re}\, z_0$; (*b*) $\lim_{z \to z_0} \overline{z} = \overline{z_0}$; (*c*) $\lim_{z \to 0} \dfrac{\overline{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 14, of limit to show that

(*a*) $\lim_{z \to z_0} (az + b) = az_0 + b$; (*b*) $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$;

(*c*) $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i \ (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 15 and limits appearing in that section to find

(*a*) $\lim_{z \to z_0} \dfrac{1}{z^n} \ (z_0 \neq 0)$; (*b*) $\lim_{z \to i} \dfrac{iz^3 - 1}{z + i}$; (*c*) $\lim_{z \to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (*a*) $1/z_0^n$; (*b*) 0; (*c*) $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 15, of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n$$

when n is a positive integer ($n = 1, 2, \ldots$).

5. Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 14.]

* See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 125–126 and p. 529, 1983.

6. Prove statement (8) in Theorem 2 of Sec. 15 using
   (*a*) Theorem 1 in Sec. 15 and properties of limits of real-valued functions of two real variables;
   (*b*) definition (2), Sec. 14, of limit.
7. Use definition (2), Sec. 14, of limit to prove that

$$\text{if} \quad \lim_{z\to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z\to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how inequality (8), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \le |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z\to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z\to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z\to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \le M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 16 to show that

$$(a) \lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4; \qquad (b) \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty; \qquad (c) \lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty.$$

11. With the aid of the theorem in Sec. 16, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \ne 0),$$

   (*a*) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;
   (*b*) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \ne 0$.
12. State why limits involving the point at infinity are unique.
13. Show that a set S is unbounded (Sec. 10) if and only if every neighborhood of the point at infinity contains at least one point in S.

18. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood of a point z_0. The *derivative* of f at z_0, written $f'(z_0)$, is defined by the equation

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

provided this limit exists. The function f is said to be *differentiable* at z_0 when its derivative at z_0 exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0,$$

we can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

Note that, because f is defined throughout a neighborhood of z_0, the number

$$f(z_0 + \Delta z)$$

is always defined for $|\Delta z|$ sufficiently small (Fig. 28).

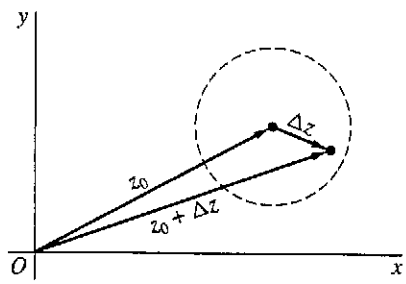

FIGURE 28

When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z),$$

which denotes the change in the value of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation (2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{3}$$

EXAMPLE 1. Suppose that $f(z) = z^2$. At any point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z,$$

since $2z + \Delta z$ is a polynomial in Δz. Hence $dw/dz = 2z$, or $f'(z) = 2z$.

EXAMPLE 2. Consider now the function $f(z) = |z|^2$. Here

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}.$$

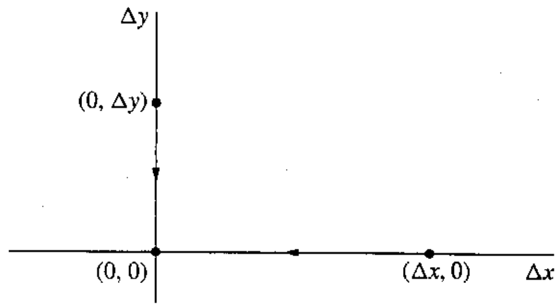

FIGURE 29

If the limit of $\Delta w/\Delta z$ exists, it may be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin in the Δz plane in any manner. In particular, when Δz approaches the origin horizontally through the points $(\Delta x, 0)$ on the real axis (Fig. 29),

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z.$$

In that case,

$$\frac{\Delta w}{\Delta z} = \overline{z} + \overline{\Delta z} + z.$$

Hence, if the limit of $\Delta w/\Delta z$ exists, its value must be $\overline{z} + z$. However, when Δz approaches the origin vertically through the points $(0, \Delta y)$ on the imaginary axis, so that

$$\overline{\Delta z} = \overline{0 + i\Delta y} = -(0 + i\Delta y) = -\Delta z,$$

we find that

$$\frac{\Delta w}{\Delta z} = \overline{z} + \overline{\Delta z} - z.$$

Hence the limit must be $\overline{z} - z$ if it exists. Since limits are unique (Sec. 14), it follows that

$$\overline{z} + z = \overline{z} - z,$$

or $z = 0$, if dw/dz is to exist.

To show that dw/dz does, in fact, exist at $z = 0$, we need only observe that our expression for $\Delta w/\Delta z$ reduces to $\overline{\Delta z}$ when $z = 0$. We conclude, therefore, that dw/dz exists *only* at $z = 0$, its value there being 0.

Example 2 shows that a function can be differentiable at a certain point but nowhere else in any neighborhood of that point. Since the real and imaginary parts of $f(z) = |z|^2$ are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0, \tag{4}$$

respectively, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point and yet the function may not be differentiable there.

The function $f(z) = |z|^2$ is continuous at each point in the plane since its components (4) are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there. It is, however, true that *the existence of the derivative of a function at a point implies the continuity of the function at that point.* To see this, we assume that $f'(z_0)$ exists and write

$$\lim_{z\to z_0}[f(z) - f(z_0)] = \lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0}\lim_{z\to z_0}(z - z_0) = f'(z_0)\cdot 0 = 0,$$

from which it follows that

$$\lim_{z\to z_0} f(z) = f(z_0).$$

This is the statement of continuity of f at z_0 (Sec. 17).

Geometric interpretations of derivatives of functions of a complex variable are not as immediate as they are for derivatives of functions of a real variable. We defer the development of such interpretations until Chap. 9.

19. DIFFERENTIATION FORMULAS

The definition of derivative in Sec. 18 is identical in form to that of the derivative of a real-valued function of a real variable. In fact, the basic differentiation formulas given below can be derived from that definition by essentially the same steps as the ones used in calculus. In these formulas, the derivative of a function f at a point z is denoted by either

$$\frac{d}{dz}f(z) \quad \text{or} \quad f'(z),$$

depending on which notation is more convenient.

Let c be a complex constant, and let f be a function whose derivative exists at a point z. It is easy to show that

$$\frac{d}{dz}c = 0, \quad \frac{d}{dz}z = 1, \quad \frac{d}{dz}[cf(z)] = cf'(z). \tag{1}$$

Also, if n is a positive integer,

$$\frac{d}{dz}z^n = nz^{n-1}. \tag{2}$$

This formula remains valid when n is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions f and F exist at a point z, then

$$\frac{d}{dz}[f(z)+F(z)] = f'(z)+F'(z), \tag{3}$$

$$\frac{d}{dz}[f(z)F(z)] = f(z)F'(z)+f'(z)F(z); \tag{4}$$

and, when $F(z) \neq 0$,

$$\frac{d}{dz}\left[\frac{f(z)}{F(z)}\right] = \frac{F(z)f'(z)-f(z)F'(z)}{[F(z)]^2}. \tag{5}$$

Let us derive formula (4). To do this, we write the following expression for the change in the product $w = f(z)F(z)$:

$$\begin{aligned}\Delta w &= f(z+\Delta z)F(z+\Delta z) - f(z)F(z)\\ &= f(z)[F(z+\Delta z)-F(z)] + [f(z+\Delta z)-f(z)]F(z+\Delta z).\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\frac{F(z+\Delta z)-F(z)}{\Delta z} + \frac{f(z+\Delta z)-f(z)}{\Delta z}F(z+\Delta z);$$

and, letting Δz tend to zero, we arrive at the desired formula for the derivative of $f(z)F(z)$. Here we have used the fact that F is continuous at the point z, since $F'(z)$ exists; thus $F(z+\Delta z)$ tends to $F(z)$ as Δz tends to zero (see Exercise 8, Sec. 17).

There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0). \tag{6}$$

If we write $w = f(z)$ and $W = g(w)$, so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}.$$

EXAMPLE. To find the derivative of $(2z^2+i)^5$, write $w = 2z^2+i$ and $W = w^5$. Then

$$\frac{d}{dz}(2z^2+i)^5 = 5w^4 4z = 20z(2z^2+i)^4.$$

To start the proof of formula (6), choose a specific point z_0 at which $f'(z_0)$ exists. Write $w_0 = f(z_0)$ and also assume that $g'(w_0)$ exists. There is, then, some ε neighborhood $|w-w_0| < \varepsilon$ of w_0 such that, for all points w in that neighborhood, we can define a function Φ which has the values $\Phi(w_0) = 0$ and

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when} \quad w \neq w_0. \tag{7}$$

Note that, in view of the definition of derivative,

$$\lim_{w \to w_0} \Phi(w) = 0. \tag{8}$$

Hence Φ is continuous at w_0.

Now expression (7) can be put in the form.

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad (|w - w_0| < \varepsilon), \tag{9}$$

which is valid even when $w = w_0$; and, since $f'(z_0)$ exists and f is, therefore, continuous at z_0, we can choose a positive number δ such that the point $f(z)$ lies in the ε neighborhood $|w - w_0| < \varepsilon$ of w_0 if z lies in the δ neighborhood $|z - z_0| < \delta$ of z_0. Thus it is legitimate to replace the variable w in equation (9) by $f(z)$ when z is any point in the neighborhood $|z - z_0| < \delta$. With that substitution, and with $w_0 = f(z_0)$, equation (9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \qquad (0 < |z - z_0| < \delta), \tag{10}$$

where we must stipulate that $z \neq z_0$ so that we are not dividing by zero. As already noted, f is continuous at z_0 and Φ is continuous at the point $w_0 = f(z_0)$. Thus the composition $\Phi[f(z)]$ is continuous at z_0; and, since $\Phi(w_0) = 0$,

$$\lim_{z \to z_0} \Phi[f(z)] = 0.$$

So equation (10) becomes equation (6) in the limit as z approaches z_0.

EXERCISES

1. Use results in Sec. 19 to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$; (b) $f(z) = (1 - 4z^2)^3$;

(c) $f(z) = \dfrac{z - 1}{2z + 1}$ $(z \neq -1/2)$; (d) $f(z) = \dfrac{(1 + z^2)^4}{z^2}$ $(z \neq 0)$.

2. Using results in Sec. 19, show that

(a) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(*b*) the coefficients in the polynomial $P(z)$ in part (*a*) can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

3. Apply definition (3), Sec. 18, of derivative to give a direct proof that

$$f'(z) = -\frac{1}{z^2} \quad \text{when} \quad f(z) = \frac{1}{z} \ (z \neq 0).$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 18, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive formula (3), Sec. 19, for the derivative of the sum of two functions.
6. Derive expression (2), Sec. 19, for the derivative of z^n when n is a positive integer by using
   (*a*) mathematical induction and formula (4), Sec. 19, for the derivative of the product of two functions;
   (*b*) definition (3), Sec. 18, of derivative and the binomial formula (Sec .3).
7. Prove that expression (2), Sec. 19, for the derivative of z^n remains valid when n is a negative integer ($n = -1, -2, \ldots$), provided that $z \neq 0$.
   Suggestion: Write $m = -n$ and use the formula for the derivative of a quotient of two functions.
8. Use the method in Example 2, Sec. 18, to show that $f'(z)$ does not exist at any point z when
   (*a*) $f(z) = \bar{z}$; (*b*) $f(z) = \text{Re}\, z$; (*c*) $f(z) = \text{Im}\, z$.
9. Let f denote the function whose values are

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

   Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\, \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. (Note that, to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane.)

20. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

(1) $$f(z) = u(x, y) + iv(x, y)$$