CHAPTER

1

COMPLEX NUMBERS

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane*, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the *real axis*, it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers*. The y axis is, then, referred to as the *imaginary axis*.

It is customary to denote a complex number (x, y) by z, so that

(1) $$z = (x, y).$$

The real numbers x and y are, moreover, known as the *real and imaginary parts* of z, respectively; and we write

(2) $$\operatorname{Re} z = x, \quad \operatorname{Im} z = y.$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts. Thus the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex, or z, plane.

The *sum* $z_1 + z_2$ and the *product* $z_1 z_2$ of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2). \tag{4}$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$
$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and, if we think of a real number as either x or $(x, 0)$ and *let i denote the pure imaginary number* $(0, 1)$ (see Fig. 1), it is clear that*

$$z = x + iy. \tag{5}$$

Also, with the convention $z^2 = zz$, $z^3 = zz^2$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

$$i^2 = -1. \tag{6}$$

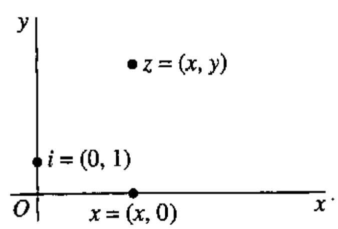

FIGURE 1

In view of expression (5), definitions (3) and (4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{8}$$

* In electrical engineering, the letter j is used instead of i.

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs.

2. BASIC ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

The commutative laws

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \tag{1}$$

and the associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3) \tag{2}$$

follow easily from the definitions in Sec. 1 of addition and multiplication of complex numbers and the fact that real numbers obey these laws. For example, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the distributive law

$$z(z_1 + z_2) = z z_1 + z z_2, \tag{3}$$

is similar.

According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, as is the case with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z \tag{4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 9).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation $(x, y) + (u, v) = (0, 0)$ implies that $u = -x$ and $v = -y$. Expression (5) can also be written $-z = -x - iy$ without ambiguity since

(Exercise 8) $-(iy) = (-i)y = i(-y)$. Additive inverses are used to define subtraction:

(6) $$z_1 - z_2 = z_1 + (-z_2).$$

So if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

(7) $$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2).$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \quad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

(8) $$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \qquad (z \neq 0).$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (8).

EXERCISES

✓ **1.** Verify that

(*a*) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$; (*b*) $(2, -3)(-2, 1) = (-1, 8)$;

(*c*) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

✓ **2.** Show that

(*a*) $\text{Re}(iz) = -\text{Im}\, z$; (*b*) $\text{Im}(iz) = \text{Re}\, z$.

✓ **3.** Show that $(1 + z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

✓✓ **5.** Prove that multiplication is commutative, as stated in the second of equations (1), Sec. 2.

6. Verify

(*a*) the associative law for addition, stated in the first of equations (2), Sec. 2;

(*b*) the distributive law (3), Sec. 2.

✓ **7.** Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. By writing $i = (0, 1)$ and $y = (y, 0)$, show that $-(iy) = (-i)y = i(-y)$.

✓✓ **9.** (*a*) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(*b*) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

✓ **10.** Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

3. FURTHER PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Sec. 2. Inasmuch as such properties continue to be anticipated because they also apply to real numbers, the reader can easily pass to Sec. 4 without serious disruption.

We begin with the observation that the existence of multiplicative inverses enables us to show that *if a product z_1z_2 is zero, then so is at least one of the factors z_1 and z_2*. For suppose that $z_1z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and, according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2 = 1 \cdot z_2 = (z_1^{-1}z_1)z_2 = z_1^{-1}(z_1z_2) = z_1^{-1} \cdot 0 = 0.$$

That is, if $z_1z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both z_1 and z_2 equal zero. Another way to state this result is that *if two complex numbers z_1 and z_2 are nonzero, then so is their product z_1z_2.*

Division by a nonzero complex number is defined as follows:

$$\frac{z_1}{z_2} = z_1z_2^{-1} \qquad (z_2 \neq 0). \tag{1}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, equation (1) here and expression (8) in Sec. 2 tell us that

$$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right).$$

That is,

$$(2)\qquad \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0).$$

Although expression (2) is not easy to remember, it can be obtained by writing (see Exercise 7)

$$(3)\qquad \frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)},$$

multiplying out the products in the numerator and denominator on the right, and then using the property

$$(4)\qquad \frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1z_3^{-1} + z_2z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \qquad (z_3 \neq 0).$$

The motivation for starting with equation (3) appears in Sec. 5.

There are some expected identities, involving quotients, that follow from the relation

$$(5)\qquad \frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0),$$

which is equation (1) when $z_1 = 1$. Relation (5) enables us, for example, to write equation (1) in the form

$$(6)\qquad \frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0).$$

Also, by observing that (see Exercise 3)

$$(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \qquad (z_1 \neq 0,\ z_2 \neq 0),$$

and hence that $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$, one can use relation (5) to show that

$$(7)\qquad \frac{1}{z_1z_2} = (z_1z_2)^{-1} = z_1^{-1}z_2^{-1} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \qquad (z_1 \neq 0,\ z_2 \neq 0).$$

Another useful identity, to be derived in the exercises, is

$$(8)\qquad \frac{z_1z_2}{z_3z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \qquad (z_3 \neq 0,\ z_4 \neq 0).$$

EXAMPLE. Computations such as the following are now justified:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right) = \frac{1}{(2-3i)(1+i)} = \frac{1}{5-i}\cdot\frac{5+i}{5+i} = \frac{5+i}{(5-i)(5+i)}$$
$$= \frac{5+i}{26} = \frac{5}{26} + \frac{i}{26} = \frac{5}{26} + \frac{1}{26}i.$$

Finally, we note that the *binomial formula* involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two complex numbers,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \qquad (n = 1, 2, \ldots) \tag{9}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof, by mathematical induction, is left as an exercise.

EXERCISES

1. Reduce each of these quantities to a real number:

$(a)\ \dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$; $\quad (b)\ \dfrac{5i}{(1-i)(2-i)(3-i)}$; $\quad (c)\ (1-i)^4$.

Ans. $(a)\ -2/5$; $\quad (b)\ -1/2$; $\quad (c)\ -4$.

2. Show that

$(a)\ (-1)z = -z$; $\quad (b)\ \dfrac{1}{1/z} = z \ (z \neq 0)$.

3. Use the associative and commutative laws for multiplication to show that

$$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4).$$

4. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1 z_2) z_3 = 0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (2), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (6) and (7) in Sec. 3, derive identity (8) there.

7. Use identity (8) in Sec. 3 to derive the cancellation law:

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0,\ z \neq 0).$$

√√**8.** Use mathematical induction to verify the binomial formula (9) in Sec. 3. More precisely, note first that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

4. MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z (Sec. 1) in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

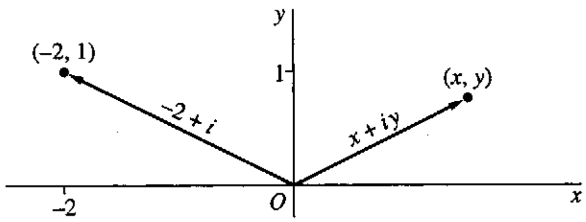

FIGURE 2

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the number $z_1 + z_2$ corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 3. The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors for z_1 and $-z_2$ (Fig. 4).

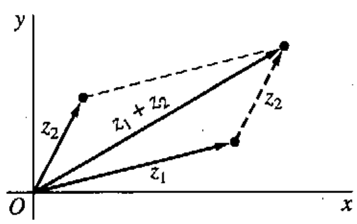

FIGURE 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus*, or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real

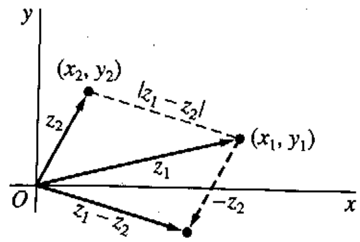

FIGURE 4

number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$(1) \qquad |z| = \sqrt{x^2 + y^2}.$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that, while *the inequality* $z_1 < z_2$ *is meaningless unless both* z_1 *and* z_2 *are real,* the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

EXAMPLE 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $|z_1 - z_2|$. This is clear from Fig. 4, since $|z_1 - z_2|$ is the length of the vector representing $z_1 - z_2$; and, by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1). Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and definition (1) that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

EXAMPLE 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1) that the real numbers $|z|$, $\text{Re } z = x$, and $\text{Im } z = y$ are related by the equation

$$(2) \qquad |z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2.$$

Thus

(3) $$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|.$$

We turn now to the *triangle inequality*, which provides an upper bound for the modulus of the sum of two complex numbers z_1 and z_2:

(4) $$|z_1 + z_2| \le |z_1| + |z_2|.$$

This important inequality is geometrically evident in Fig. 3, since it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We can also see from Fig. 3 that inequality (4) is actually an equality when 0, z_1, and z_2 are collinear. Another, strictly algebraic, derivation is given in Exercise 16, Sec. 5.

An immediate consequence of the triangle inequality is the fact that

(5) $$|z_1 + z_2| \ge ||z_1| - |z_2||.$$

To derive inequality (5), we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |-z_2|,$$

which means that

(6) $$|z_1 + z_2| \ge |z_1| - |z_2|.$$

This is inequality (5) when $|z_1| \ge |z_2|$. If $|z_1| < |z_2|$, we need only interchange z_1 and z_2 in inequality (6) to get

$$|z_1 + z_2| \ge -(|z_1| - |z_2|),$$

which is the desired result. Inequality (5) tells us, of course, that the length of one side of a triangle is greater than or equal to the difference of the lengths of the other two sides.

Because $|-z_2| = |z_2|$, one can replace z_2 by $-z_2$ in inequalities (4) and (5) to summarize these results in a particularly useful form:

(7) $$|z_1 \pm z_2| \le |z_1| + |z_2|,$$

(8) $$|z_1 \pm z_2| \ge ||z_1| - |z_2||.$$

EXAMPLE 3. If a point z lies on the unit circle $|z| = 1$ about the origin, then

$$|z - 2| \le |z| + 2 = 3$$

and

$$|z - 2| \ge ||z| - 2| = 1.$$

The triangle inequality (4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$(9) \qquad |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots).$$

To give details of the induction proof here, we note that when $n = 2$, inequality (9) is just inequality (4). Furthermore, if inequality (9) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by inequality (4),

$$|(z_1 + z_2 + \cdots + z_m) + z_{m+1}| \le |z_1 + z_2 + \cdots + z_m| + |z_{m+1}|$$
$$\le (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|.$$

EXERCISES

✓ **1.** Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(*a*) $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$; (*b*) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

(*c*) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; (*d*) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

✓✓ **2.** Verify inequalities (3), Sec. 4, involving $\text{Re } z$, $\text{Im } z$, and $|z|$.

✓✓ **3.** Verify that $\sqrt{2}\,|z| \ge |\text{Re } z| + |\text{Im } z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \ge 0$.

✓**4.** In each case, sketch the set of points determined by the given condition:

(*a*) $|z - 1 + i| = 1$; (*b*) $|z + i| \le 3$; (*c*) $|z - 4i| \ge 4$.

✓**5.** Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(*a*) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

(*b*) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

5. COMPLEX CONJUGATES

The *complex conjugate*, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

$$(1) \qquad \bar{z} = x - iy.$$

The number $\bar{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 5). Note that

$$\bar{\bar{z}} = z \quad \text{and} \quad |\bar{z}| = |z|$$

for all z.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

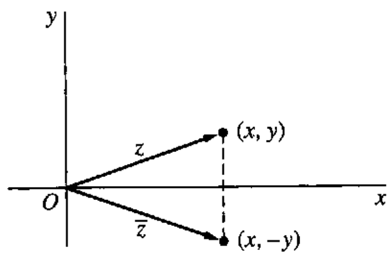

FIGURE 5

So the conjugate of the sum is the sum of the conjugates:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}. \tag{2}$$

In like manner, it is easy to show that

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}, \tag{3}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \tag{4}$$

and

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0). \tag{5}$$

The sum $z + \overline{z}$ of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ is the real number $2x$, and the difference $z - \overline{z}$ is the pure imaginary number $2iy$. Hence

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}. \tag{6}$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\overline{z} = |z|^2, \tag{7}$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (3), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 1. As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i.$$

See also the example near the end of Sec. 3.

Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

$$|z_1z_2| = |z_1||z_2| \tag{8}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{9}$$

Property (8) can be established by writing

$$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) = (z_1z_2)(\overline{z_1}\,\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized form (9) of the triangle inequality in Sec. 4 that

$$|z^3 + 3z^2 - 2z + 1| \leq |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Use properties of conjugates and moduli established in Sec. 5 to show that
   (*a*) $\overline{\bar{z} + 3i} = z - 3i$; (*b*) $\overline{iz} = -i\bar{z}$;
   (*c*) $\overline{(2+i)^2} = 3 - 4i$; (*d*) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}\,|2z + 5|$.
2. Sketch the set of points determined by the condition
   (*a*) $\text{Re}(\bar{z} - i) = 2$; (*b*) $|2\bar{z} - i| = 4$.
3. Verify properties (3) and (4) of conjugates in Sec. 5.
4. Use property (4) of conjugates in Sec. 5 to show that
   (*a*) $\overline{z_1z_2z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; (*b*) $\overline{z^4} = \bar{z}^4$.
5. Verify property (9) of moduli in Sec. 5.
6. Use results in Sec. 5 to show that when z_2 and z_3 are nonzero,
   (*a*) $\overline{\left(\dfrac{z_1}{z_2z_3}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}\,\overline{z_3}}$; (*b*) $\left|\dfrac{z_1}{z_2z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.
7. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,
$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

8. Show that

$$|\mathrm{Re}(2 + \overline{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

9. It is shown in Sec. 3 that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 5.

10. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and then using inequality (8), Sec. 4, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

11. Prove that

(*a*) z is real if and only if $\overline{z} = z$;

(*b*) z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

12. Use mathematical induction to show that when $n = 2, 3, \ldots,$

(*a*) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (*b*) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

13. Let $a_0, a_1, a_2, \ldots, a_n$ $(n \ge 1)$ denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 12, show that

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^n.$$

14. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

15. Using expressions (6), Sec. 5, for $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

16. Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 4)

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

(*a*) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

(*b*) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \le 2|z_1||z_2|.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the inequality

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

6. EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in *polar form* as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined; and so it is always understood that $z \neq 0$ whenever θ is discussed.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\operatorname{Arg} z$, is that unique value Θ such that $-\pi < \Theta \leq \pi$. Note that

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Also, when z is a negative real number, $\operatorname{Arg} z$ has value π, not $-\pi$.

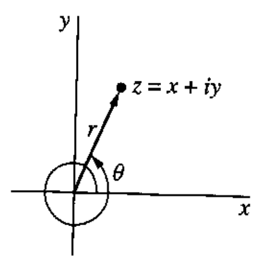

FIGURE 6

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}.$$

It must be emphasized that, because of the restriction $-\pi < \Theta \leq \pi$ of the principal argument Θ, it is *not* true that $\operatorname{Arg}(-1 - i) = 5\pi/4$.

According to equation (2),

$$\arg(-1 - i) = -\frac{3\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that the term Arg z on the right-hand side of equation (2) can be replaced by any particular value of arg z and that one can write, for instance,

$$\arg(-1-i) = \frac{5\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The symbol $e^{i\theta}$, or $\exp(i\theta)$, is defined by means of *Euler's formula* as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

where θ is to be measured in radians. It enables us to write the polar form (1) more compactly in *exponential form* as

$$z = re^{i\theta}. \tag{4}$$

The choice of the symbol $e^{i\theta}$ will be fully motivated later on in Sec. 28. Its use in Sec. 7 will, however, suggest that it is a natural choice.

EXAMPLE 2. The number $-1-i$ in Example 1 has exponential form

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \tag{5}$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written $-1-i = \sqrt{2}\,e^{-i3\pi/4}$. Expression (5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

Note how expression (4) with $r = 1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in Fig. 7. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance,

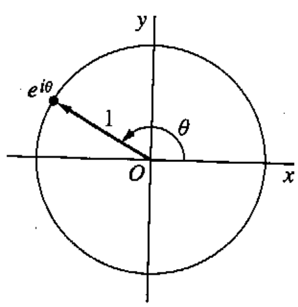

FIGURE 7

geometrically obvious that

$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad \text{and} \quad e^{-i4\pi} = 1.$$

Note, too, that the equation

$$z = Re^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is a parametric representation of the circle $|z| = R$, centered at the origin with radius R. As the parameter θ increases from $\theta = 0$ to $\theta = 2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi). \tag{8}$$

This can be seen vectorially (Fig. 8) by noting that a point z traversing the circle $|z - z_0| = R$ once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

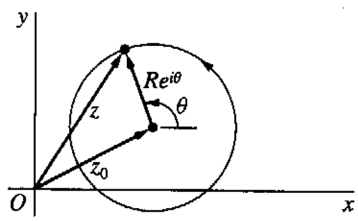

FIGURE 8

7. PRODUCTS AND QUOTIENTS IN EXPONENTIAL FORM

Simple trigonometry tells us that $e^{i\theta}$ has the familiar additive property of the exponential function in calculus:

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

Thus, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, the product z_1z_2 has exponential form

$$z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}. \tag{1}$$

Moreover,

(2) $$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \frac{e^{i\theta_1}e^{-i\theta_2}}{e^{i\theta_2}e^{-i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

Because $1 = 1e^{i0}$, it follows from expression (2) that the inverse of any nonzero complex number $z = re^{i\theta}$ is

(3) $$z^{-1} = \frac{1}{z} = \frac{1}{r}e^{-i\theta}.$$

Expressions (1), (2), and (3) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x.

Expression (1) yields an important identity involving arguments:

(4) $$\arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

It is to be interpreted as saying that if values of two of these three (multiple-valued) arguments are specified, then there is a value of the third such that the equation holds.

We start the verification of statement (4) by letting θ_1 and θ_2 denote any values of $\arg z_1$ and $\arg z_2$, respectively. Expression (1) then tells us that $\theta_1 + \theta_2$ is a value of $\arg(z_1 z_2)$. (See Fig. 9.) If, on the other hand, values of $\arg(z_1 z_2)$ and $\arg z_1$ are specified, those values correspond to particular choices of n and n_1 in the expressions

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\arg z_1 = \theta_1 + 2n_1\pi \qquad (n_1 = 0, \pm 1, \pm 2, \ldots).$$

Since

$$(\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + [\theta_2 + 2(n - n_1)\pi],$$

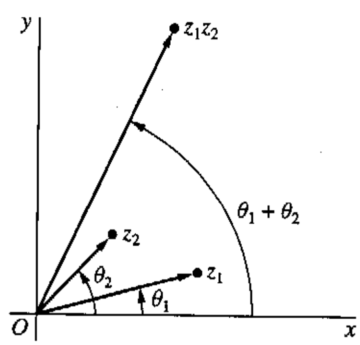

FIGURE 9

equation (4) is evidently satisfied when the value

$$\arg z_2 = \theta_2 + 2(n - n_1)\pi$$

is chosen. Verification when values of $\arg(z_1 z_2)$ and $\arg z_2$ are specified follows by symmetry.

Statement (4) is sometimes valid when *arg* is replaced everywhere by *Arg* (see Exercise 7). But, as the following example illustrates, that is *not always* the case.

EXAMPLE 1. When $z_1 = -1$ and $z_2 = i$,

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \quad \text{but} \quad \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

If, however, we take the values of $\arg z_1$ and $\arg z_2$ just used and select the value

$$\operatorname{Arg}(z_1 z_2) + 2\pi = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$$

of $\arg(z_1 z_2)$, we find that equation (4) *is* satisfied.

Statement (4) tells us that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 z_2^{-1}) = \arg z_1 + \arg(z_2^{-1}),$$

and we can see from expression (3) that

$$\arg(z_2^{-1}) = -\arg z_2. \tag{5}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{6}$$

Statement (5) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (6) is, then, to be interpreted in the same way that statement (4) is.

EXAMPLE 2. In order to find the principal argument Arg z when

$$z = \frac{-2}{1 + \sqrt{3}i},$$

observe that

$$\arg z = \arg(-2) - \arg(1 + \sqrt{3}i).$$

Since

$$\operatorname{Arg}(-2) = \pi \quad \text{and} \quad \operatorname{Arg}(1 + \sqrt{3}i) = \frac{\pi}{3},$$

one value of $\arg z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and π, we find that $\operatorname{Arg} z = 2\pi/3$.

Another important result that can be obtained formally by applying rules for real numbers to $z = re^{i\theta}$ is

$$(7) \qquad z^n = r^n e^{in\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

It is easily verified for positive values of n by mathematical induction. To be specific, we first note that it becomes $z = re^{i\theta}$ when $n = 1$. Next, we assume that it is valid when $n = m$, where m is any positive integer. In view of expression (1) for the product of two nonzero complex numbers in exponential form, it is then valid for $n = m + 1$:

$$z^{m+1} = zz^m = re^{i\theta} r^m e^{im\theta} = r^{m+1} e^{i(m+1)\theta}.$$

Expression (7) is thus verified when n is a positive integer. It also holds when $n = 0$, with the convention that $z^0 = 1$. If $n = -1, -2, \ldots$, on the other hand, we define z^n in terms of the multiplicative inverse of z by writing

$$z^n = (z^{-1})^m \quad \text{where} \quad m = -n = 1, 2, \ldots.$$

Then, since expression (7) is valid for positive integral powers, it follows from the exponential form (3) of z^{-1} that

$$z^n = \left[\frac{1}{r} e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{in\theta} \qquad (n = -1, -2, \ldots).$$

Expression (7) is now established for all integral powers.

Observe that if $r = 1$, expression (7) becomes

$$(8) \qquad (e^{i\theta})^n = e^{in\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

When written in the form

$$(9) \qquad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

this is known as *de Moivre's formula.*

Expression (7) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

EXAMPLE 3. In order to put $(\sqrt{3}+i)^7$ in rectangular form, one need only write

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3}+i).$$

EXERCISES

✓ **1.** Find the principal argument Arg z when

$(a)\ z = \dfrac{i}{-2-2i}$; $\quad (b)\ z = (\sqrt{3}-i)^6$.

Ans. $(a)\ -3\pi/4$; $\quad (b)\ \pi$.

✓ **2.** Show that $(a)\ |e^{i\theta}| = 1$; $\quad (b)\ \overline{e^{i\theta}} = e^{-i\theta}$.

✓✓ **3.** Use mathematical induction to show that

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

4. Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

Ans. π.

✓ **5.** Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

$(a)\ \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\quad (b)\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

✓ **6.** By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

$(a)\ i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; $\quad (b)\ 5i/(2+i) = 1+2i$;

$(c)\ (-1+i)^7 = -8(1+i)$; $\quad (d)\ (1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

✓✓ **7.** Show that if Re $z_1 > 0$ and Re $z_2 > 0$, then

$$\operatorname{Arg}(z_1z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where $\operatorname{Arg}(z_1z_2)$ denotes the principal value of $\arg(z_1z_2)$, etc.

8. Let z be a nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right)e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

9. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 2(*b*)]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

10. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

11. (*a*) Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i \sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and use the above sum to obtain the expression [compare Exercise 5(*a*)]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta \qquad (n = 1, 2, \ldots).$$

(*b*) Write $x = \cos\theta$ and suppose that $0 \leq \theta \leq \pi$, in which case $-1 \leq x \leq 1$. Point out how it follows from the final result in part (*a*) that each of the functions

$$T_n(x) = \cos(n \cos^{-1} x) \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n in the variable x.*

8. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is

* These polynomials are called Chebyshev polynomials and are prominent in approximation theory.

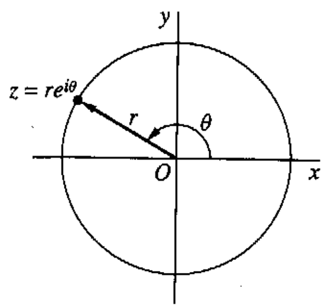

FIGURE 10

true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \textit{and} \quad \theta_1 = \theta_2 + 2k\pi,$$

where k is some integer $(k = 0, \pm 1, \pm 2, \ldots)$.

This observation, together with the expression $z^n = r^n e^{in\theta}$ in Sec. 7 for integral powers of complex numbers $z = re^{i\theta}$, is useful in finding the nth roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n = 2, 3, \ldots$. The method starts with the fact that an nth root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

According to the statement in italics just above, then,

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi,$$

where k is any integer $(k = 0, \pm 1, \pm 2, \ldots)$. So $r = \sqrt[n]{r_0}$, where this radical denotes the unique *positive* nth root of the positive real number r_0, and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

are the nth roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument θ_0/n. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n-1$, and no further roots arise with other values of k. We let c_k $(k = 0, 1, 2, \ldots, n-1)$ denote these distinct roots and write

$$c_k = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n-1). \tag{1}$$

(See Fig. 11.)

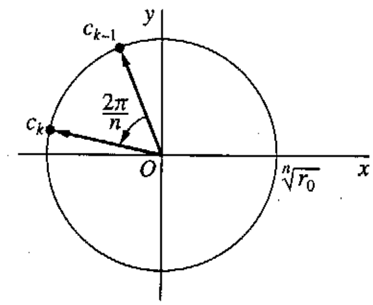

FIGURE 11

The number $\sqrt[n]{r_0}$ is the length of each of the radius vectors representing the n roots. The first root c_0 has argument θ_0/n; and the two roots when $n = 2$ lie at the opposite ends of a diameter of the circle $|z| = \sqrt[n]{r_0}$, the second root being $-c_0$. When $n \geq 3$, the roots lie at the vertices of a regular polygon of n sides inscribed in that circle.

We shall let $z_0^{1/n}$ denote the *set* of nth roots of z_0. If, in particular, z_0 is a positive real number r_0, the symbol $r_0^{1/n}$ denotes the entire set of roots; and the symbol $\sqrt[n]{r_0}$ in expression (1) is reserved for the one positive root. When the value of θ_0 that is used in expression (1) is the principal value of $\arg z_0$ $(-\pi < \theta_0 \leq \pi)$, the number c_0 is referred to as the *principal root*. Thus when z_0 is a positive real number r_0, its principal root is $\sqrt[n]{r_0}$.

Finally, a convenient way to remember expression (1) is to write z_0 in its most general exponential form (compare Example 2 in Sec. 6)

$$z_0 = r_0 e^{i(\theta_0 + 2k\pi)} \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2}$$

and to *formally* apply laws of fractional exponents involving real numbers, keeping in mind that there are precisely n roots:

$$z_0^{1/n} = \left[r_0 e^{i(\theta_0+2k\pi)}\right]^{1/n} = \sqrt[n]{r_0}\exp\left[\frac{i(\theta_0 + 2k\pi)}{n}\right]$$
$$= \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n-1).$$

The examples in the next section serve to illustrate this method for finding roots of complex numbers.

9. EXAMPLES

In each of the examples here, we start with expression (2), Sec. 8, and proceed in the manner described at the end of that section.

EXAMPLE 1. In order to determine the nth roots of unity, we write

$$1 = 1\exp[i(0 + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2 \ldots)$$

and find that

$$(1)\quad 1^{1/n} = \sqrt[n]{1}\exp\left[i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1).$$

When $n = 2$, these roots are, of course, ± 1. When $n \geq 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z| = 1$, with one vertex corresponding to the principal root $z = 1$ $(k = 0)$.

If we write

$$(2)\qquad \omega_n = \exp\left(i\frac{2\pi}{n}\right),$$

it follows from property (8), Sec. 7, of $e^{i\theta}$ that

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1).$$

Hence the distinct nth roots of unity just found are simply

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.$$

See Fig. 12, where the cases $n = 3$, 4, and 6 are illustrated. Note that $\omega_n^n = 1$. Finally,

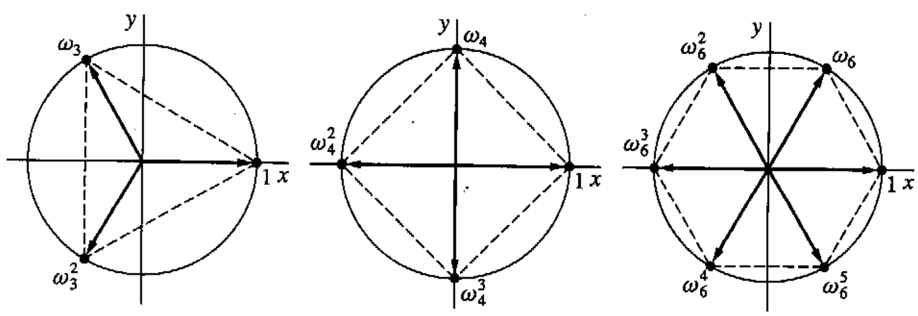

FIGURE 12

it is worthwhile observing that if c is any particular nth root of a nonzero complex number z_0, the set of nth roots can be put in the form

$$c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}.$$

This is because multiplication of any nonzero complex number by ω_n increases the argument of that number by $2\pi/n$, while leaving its modulus unchanged.

EXAMPLE 2. Let us find all values of $(-8i)^{1/3}$, or the three cube roots of $-8i$. One need only write

$$-8i = 8\exp\left[i\left(-\frac{\pi}{2} + 2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

to see that the desired roots are

$$c_k = 2\exp\left[i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2). \tag{3}$$

They lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$, and are equally spaced around that circle every $2\pi/3$ radians, starting with the principal root (Fig. 13)

$$c_0 = 2\exp\left[i\left(-\frac{\pi}{6}\right)\right] = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{3} - i.$$

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} - i$.

These roots can, of course, be written

$$c_0, c_0\omega_3, c_0\omega_3^2 \quad \text{where} \quad \omega_3 = \exp\left(i\frac{2\pi}{3}\right).$$

(See the remarks at the end of Example 1.)

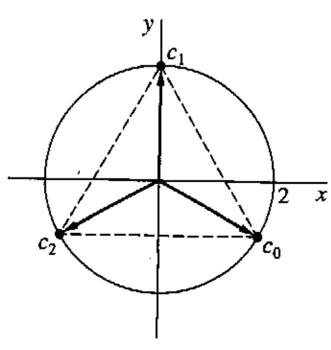

FIGURE 13

EXAMPLE 3. The two values c_k $(k = 0, 1)$ of $(\sqrt{3}+i)^{1/2}$, which are the square roots of $\sqrt{3}+i$, are found by writing

$$\sqrt{3}+i = 2\exp\left[i\left(\frac{\pi}{6}+2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

and (see Fig. 14)

$$c_k = \sqrt{2}\exp\left[i\left(\frac{\pi}{12}+k\pi\right)\right] \qquad (k = 0, 1). \tag{4}$$

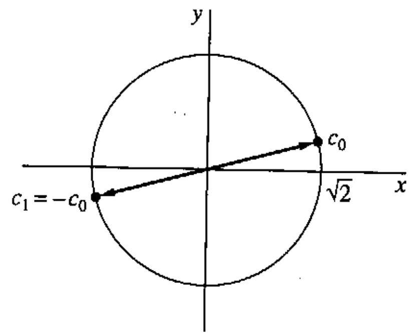

FIGURE 14

Euler's formula (Sec. 6) tells us that

$$c_0 = \sqrt{2}\exp\left(i\frac{\pi}{12}\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),$$

and the trigonometric identities

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}, \quad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2} \tag{5}$$

enable us to write

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4},$$

$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}} + i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3}+i$ are, then,

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

EXERCISES

✓ **1.** Find the square roots of (*a*) $2i$; (*b*) $1-\sqrt{3}i$ and express them in rectangular coordinates.

Ans. (*a*) $\pm(1+i)$; (*b*) $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

✓ **2.** In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:
(*a*) $(-16)^{1/4}$; (*b*) $(-8-8\sqrt{3}i)^{1/4}$.

Ans. (*a*) $\pm\sqrt{2}(1+i), \pm\sqrt{2}(1-i)$; (*b*) $\pm(\sqrt{3}-i), \pm(1+\sqrt{3}i)$.

3. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:
(*a*) $(-1)^{1/3}$; (*b*) $8^{1/6}$.

Ans. (*b*) $\pm\sqrt{2}, \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}, \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

✓ **4.** According to Example 1 in Sec. 9, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$, where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2}+4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

5. (*a*) Let a denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right),$$

where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

(*b*) With the aid of the trigonometric identities (5) in Example 3 of Sec. 9, show that the square roots obtained in part (*a*) can be written

$$\pm\frac{1}{\sqrt{2}}(\sqrt{A+a}+i\sqrt{A-a}).$$

[Note that this becomes the final result in Example 3, Sec. 9, when $a=\sqrt{3}$.]

✓ **6.** Find the four roots of the equation $z^4+4=0$ and use them to factor z^4+4 into quadratic factors with real coefficients.

Ans. $(z^2+2z+2)(z^2-2z+2)$.

✓✓ **7.** Show that if c is any nth root of unity other than unity itself, then

$$1+c+c^2+\cdots+c^{n-1}=0.$$

Suggestion: Use the first identity in Exercise 10, Sec. 7.

✓ **8.** (*a*) Prove that the usual formula solves the quadratic equation

$$az^2+bz+c=0 \qquad (a\neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the *quadratic formula*

$$z=\frac{-b+(b^2-4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2-4ac\neq 0$,

(*b*) Use the result in part (*a*) to find the roots of the equation $z^2+2z+(1-i)=0$.

Ans. (*b*) $\left(-1+\frac{1}{\sqrt{2}}\right)+\frac{i}{\sqrt{2}},\ \left(-1-\frac{1}{\sqrt{2}}\right)-\frac{i}{\sqrt{2}}.$

9. Let $z=re^{i\theta}$ be any nonzero complex number and n a negative integer ($n=-1,-2,\ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n}=(z^{-1})^{1/m}$, where $m=-n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n}=(z^{1/m})^{-1}$. (Compare Exercise 8, Sec. 7.)

10. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the z plane, and their closeness to one another. Our basic tool is the concept of an ε *neighborhood*

$$|z-z_0|<\varepsilon \tag{1}$$

of a given point z_0. It consists of all points z lying inside but not on a circle centered at

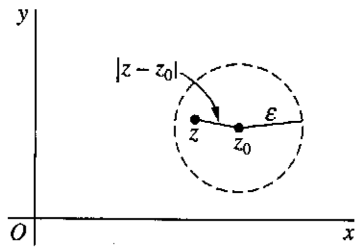

FIGURE 15

z_0 and with a specified positive radius ε (Fig. 15). When the value of ε is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*

$$0 < |z - z_0| < \varepsilon, \tag{2}$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

A point z_0 is said to be an *interior point* of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a *boundary point* of S. A boundary point is, therefore, a point all of whose neighborhoods contain points in S and points not in S. The totality of all boundary points is called the *boundary* of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

$$|z| < 1 \quad \text{and} \quad |z| \le 1. \tag{3}$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points; and the *closure* of a set S is the closed set consisting of all points in S together with the boundary of S. Note that the first of the sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the *punctured disk* $0 < |z| \le 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 16). An open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

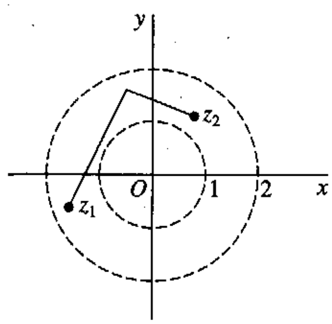

FIGURE 16

A set S is *bounded* if every point of S lies inside some circle $|z| = R$; otherwise, it is *unbounded.* Both of the sets (3) are bounded regions, and the half plane $\text{Re}\, z \geq 0$ is unbounded.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation point z_0 were not in S, it would be a boundary point of S; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus, a set is closed if and only if it contains all of its accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain points of S. Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.

EXERCISES

✓ **1.** Sketch the following sets and determine which are domains:

(*a*) $|z - 2 + i| \leq 1$; (*b*) $|2z + 3| > 4$;
(*c*) $\text{Im}\, z > 1$; (*d*) $\text{Im}\, z = 1$;
(*e*) $0 \leq \arg z \leq \pi/4$ $(z \neq 0)$; (*f*) $|z - 4| \geq |z|$.

Ans. (*b*), (*c*) are domains.

✓ **2.** Which sets in Exercise 1 are neither open nor closed?

Ans. (*e*).

✓ **3.** Which sets in Exercise 1 are bounded?

Ans. (*a*).

✓ **4.** In each case, sketch the closure of the set:

(*a*) $-\pi < \arg z < \pi$ $(z \neq 0)$; (*b*) $|\text{Re}\, z| < |z|$;
(*c*) $\text{Re}\left(\dfrac{1}{z}\right) \leq \dfrac{1}{2}$; (*d*) $\text{Re}(z^2) > 0$.

5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. State why S is not connected.

6. Show that a set S is open if and only if each point in S is an interior point.

7. Determine the accumulation points of each of the following sets:

   (a) $z_n = i^n$ $(n = 1, 2, \ldots)$; (b) $z_n = i^n/n$ $(n = 1, 2, \ldots)$;

   (c) $0 \leq \arg z < \pi/2$ $(z \neq 0)$; (d) $z_n = (-1)^n(1+i)\dfrac{n-1}{n}$ $(n = 1, 2, \ldots)$.

   Ans. (a) None; (b) 0; (d) $\pm(1+i)$.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

9. Show that any point z_0 of a domain is an accumulation point of that domain.

10. Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.