Name ______________________________

MATH 301 Differential Equations Fall 2006 Final Exam

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

You can use integration aids such as a table of integrals.

Give all solutions in terms of real-valued functions. Give explicit solutions when feasible and implicit solutions otherwise.

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1. Consider the initial-value problem $ty'(t) = y(t) + ty(t)$ with $y(1) = 3$.
   (a) Show that there is a unique solution for this initial-value problem. What is strongest statement about the t-interval on which the unique solution exists that can made without first finding the solution? (4 points)
   (b) Find the specific solution of the initial-value problem. (10 points)

2. Consider the differential equation $y''(t) + 4y'(t) - y(t) = 0$.
   (a) Find the general solution. (10 points)
   (b) Describe the behavior of solutions as $t \to \infty$. That is, do solutions have a limit? Are solutions unbounded? Are solutions oscillatory? Do solutions have different behaviors for different initial conditions? (4 points)

3. Find the general solution of $y'' - 2y' - 8y = \sin(t)$. (14 points)

4. Find the general solution of the differential equation $\dfrac{dz}{dx} = -\dfrac{z^2}{2 + xz}$.
   Hint: Try $\mu = ze^{xz}$ as an integrating factor. (8 points)

5. Determine if
$$y(t) = c_1 \frac{1}{t^2} + c_2 t^3 + t^3 \ln t$$
is the general solution of the differential equation
$$t^2 y'' - 6y = 5t^3$$
for the interval $(0, \infty)$. *You do not need to show where this formula for $y(t)$ comes from.* (8 points)

There is more on the flip side.

6. Consider the system $\dfrac{d\vec{y}}{dt} = A\vec{y}$ where for each of the following $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

(a) Find the general solution. (10 points)

(b) Sketch a phase portrait for the system. (4 points)

7. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= xy - 2y \\ \frac{dy}{dt} &= x\sin(\pi y) \end{aligned}$$

(a) Show that $(2, -1)$ is an equilibrium point for this system. (4 points)

(b) Determine if $(2, -1)$ is stable or unstable. (8 points)

8. Let x and y be measures of two interacting species. Consider the model

$$\begin{aligned} \frac{dx}{dt} &= (1 - x)x + xy \\ \frac{dy}{dt} &= (1 - y)y + axy \end{aligned}$$

where a is a positive constant.

(a) In this model, what real-world assumption is made about how each species changes in the absence of the other species? (2 points)

(b) In this model, what real-world assumption is made about the nature of the interaction between the species? (2 points)

(c) For $0 < a < 1$, use nullclines to split the first quadrant of the xy-plane into regions and indicate the general nature of direction vectors for each region. (4 points)

(d) For $a \geq 1$, use nullclines to split the first quadrant of the xy-plane into regions and indicate the general nature of direction vectors for each region. (4 points)

(e) Give a real-world interpretation of your results from (c) and (d). (4 points)