Name ______________________________

MATH 301 Differential Equations Fall 2006 Exam #3

Instructions:

Do your own work. You may consult your class notes and the course text. Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me.

Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning). If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong or on alternate approaches you might try.

You can use technology for calculations as needed with the exception of the `DSolve` command in *Mathematica* (or its equivalent in other computing technology). When you do, give some indication of what you use and how you use it.

Express each result in terms of real-valued functions.

Each problem has a maximum value of 25 points.

The exam is due Thursday, November 30.

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1. Find the general solution of the system $\frac{d}{dt}\vec{y} = A\vec{y}$ where $A = \begin{bmatrix} 20 & -8 & 16 \\ -28 & 18 & -24 \\ -44 & 22 & -36 \end{bmatrix}$.

   Describe the behavior of solutions as $t \to \infty$.

2. Consider the system $\frac{d}{dt}\vec{y} = A\vec{y}$ where $A = \begin{bmatrix} -3 & -\mu \\ \mu & 1 \end{bmatrix}$. Note that the form of the general solution, and therefore the behavior of solutions, will depend on the parameter μ. Find all different forms of the general solution. For each form, give the relevant value or values of μ. For each form of solution, make a phase portrait showing solution curves in the y_1y_2-plane. Draw each phase portrait with enough detail to show the important qualitative features of solutions.

3. Find the solution solution of the initial value problem $\frac{d}{dt}\vec{y} = A\vec{y} + \vec{g}(t)$ where

$$A = \begin{bmatrix} -5 & 1 \\ 2 & -4 \end{bmatrix} \qquad \text{and} \qquad \vec{g}(t) = e^{-2t}\begin{bmatrix} e^{-4t} \\ \sin t \end{bmatrix} \qquad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   Plot each component of the solution as a function of t.

4. Do either one of the following two problems.

   (A) Consider a chain of radioactive decays $X \to Y \to Z$ where X, Y, and Z are different radioactive elements with the decay rate of X less than the decay rate of Y which in turn is less than the decay rate of Z. Let $x(t)$, $y(t)$, and $z(t)$ denote the amounts of these elements. Set up and solve an initial value problem to determine $x(t)$, $y(t)$, and $z(t)$ for a given sample if the process starts with an initial amount x_0 of element X and none of elements Y and Z. Find expressions for the ratios $y(t)/x(t)$ and $z(t)/x(t)$ and the limits of these ratios as $t \to \infty$. Discuss how you could use these results to get information on the decay rates if you could only make measurements on the sample a long time after $t = 0$ (where long means in comparison to the half-lifes).

   (B) Consider a system of three identical blocks in a line on a frictionless surface connected to each other and fixed endwalls by four identical springs. Set up and analyze a model to determine the normal modes of vibration for this system. For each normal mode,

   - determine the frequency of the mode
   - give a set of initial conditions that will result in motion for that mode.
   - describe the motion in words and/or pictures