Name _____

MATH 301 Differential Equations Spring 2006 Exam #1

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

You can use integration aids such as a table of integrals.

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1. Consider the differential equation $\dfrac{dy}{dt} = y(4 - y^2)$.

   (a) Sketch a slope field with enough detail to show all interesting features for $-5 \le y \le 5$. (8 points)

   (b) On your slope field, sketch solution curves. Include enough to see all possible types of solution behavior. For each type, give the relevant initial values $y(0) = y_0$ and briefly describe the main features. (10 points)

2. For each of the following, solve the given differential equation or initial-value problem. (16 points each)

   (a) $\dfrac{dy}{dt} = \dfrac{y + \cos y + \cos t}{t \sin y - 2y - t}$

   (b) $\dfrac{dy}{dt} - ty^2 = t, \quad y(2) = 0$

   (c) $\dfrac{dy}{dt} = (t + y)^2 - 1, \quad y(1) = 3$ Hint: Try the substitution $z = t + y$.

3. Show that the initial-value problem

$$\frac{dy}{dt} + p(t)y = g(t) \qquad y(t_0) = y_0$$

   has the specific solution

$$y(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{P(s)} g(s)\, ds \qquad \text{where } P(t) = \int_{t_0}^{t} p(s)\, ds.$$

   Do this directly by showing $y(t)$ as given satisfies the differential equation and the initial condition. Give enough detail to make clear that you know what is going on. (10 points)

There is more on the flip side.

4. One solution of the initial-value problem

$$(ty^3)y' = 2y^4 + t^4, \qquad y(1) = 2$$

is $y(t) = \left(17t^8 - t^4\right)^{1/4}$. Show that this is the only solution of the initial-value problem. Note: You don't need to show how to get this solution. Just show that it must be the only solution. (10 points)

5. An aquarium that starts out empty is being filled with impure water. Simultaneously, a filter system is extracting water from the tank, removing a fixed percentage of the impurities, and returning the resulting water to the tank. Set up and analyze a model with the goal of determining the impurity concentration at the time the tank is filled. (14 points)