Trigonometry Review

Version 0.1 (September 6, 2004)

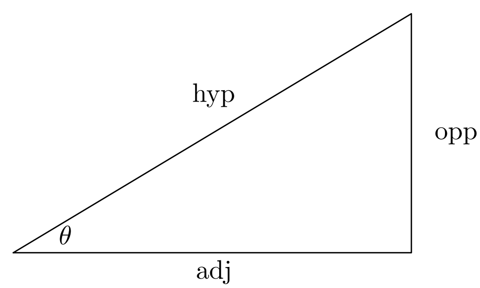
Martin Jackson, University of Puget Sound

The purpose of these notes is to provide a brief review of trigonometry for students who are taking calculus. The choice of topics is driven by the ways in which we will use trigonometry in calculus. You should work each exercise as you come to it. There is some intentional repetition in the exercises to force familiarity with some basic things.

Trigonometric functions for right triangles

The trigonometric functions are first defined as ratios of side lengths of a right triangle:

$$\begin{aligned} \sin\theta &= \frac{\text{opp}}{\text{hyp}} \\ \cos\theta &= \frac{\text{adj}}{\text{hyp}} \\ \tan\theta &= \frac{\text{opp}}{\text{adj}} \end{aligned}$$



The values of trigonometric functions depend on the angle θ but not on the size of the right triangle. If all of the side lengths are scaled by the same factor, the angles do not change and the ratios of side lengths do not change.

We can easily determine values of the trigonometric functions for some special angles.

Exercise 1. Draw a $45°$-$45°$-$90°$ triangle. Choose a side length of 1 for the two congruent sides. Determine the length of the hypotenuse. Use this triangle to find $\sin 45°$, $\cos 45°$, and $\tan 45°$.

Exercise 2. Draw a $30°$-$60°$-$90°$ triangle. Choose a side length of 1 for the short side adjacent to the right angle. Determine the length of the hypotenuse and of the other side adjacent to the right angle. (Hint: Make an equilateral triangle by reflecting the $30°$-$60°$-$90°$ triangle across the long side adjacent to the right angle.) Use this triangle to find $\sin 30°$, $\cos 30°$, and $\tan 30°$. Also use this triangle to find $\sin 60°$, $\cos 60°$, and $\tan 60°$.

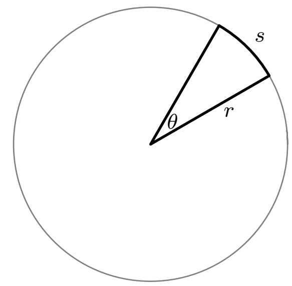
Exercise 3. Fill in the values in the table using your results from the previous two exercises.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$			
$45°$			
$60°$			

Radian measure of angles

In calculus, it is best to use radian measure of angles. Radian measure is defined by looking at the arclength s subtended by the angle in a circle of radius r. Radian measure is the ratio of arclength to radius:

$$\theta = \frac{s}{r}$$



Note that if arclength and radius are measured in the same units (inches for example), the radian measure of an angle is a unitless quantity.

Exercise 4. Determine the radian measure of the angle that subtends a full circle. Hint: In this case, the arclength subtended by the angle is the full circumference of the circle. Use the formula for the circumference of a circle of radius r.

The answer for the the previous exercise gives a conversion factor between radian measure and degree measure. The correct radian measure of the angle subtending a full circle is 2π so we have $2\pi = 360°$.

Exercise 5. Fill in the values in the table using the conversion factor. In the middle row, give the radian measure as a multiple of π. In the bottom row, give a decimal approximation of the radian measure (using your calculator).

degree measure	0°	30°	45°	60°	90°	120°	135°	150°	180°
radian measure									
decimal approx.									

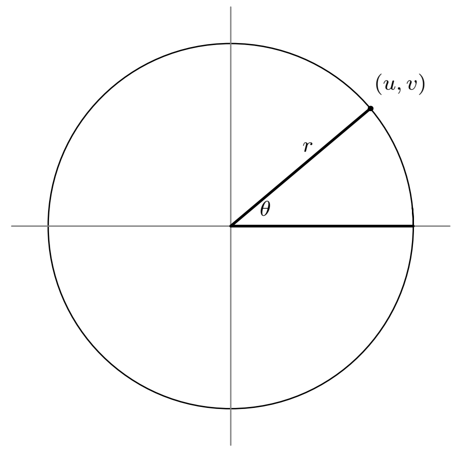
degree measure		210°	225°	240°	270°	300°	315°	330°	360°
radian measure									
decimal approx.									

Trigonometric functions and circles

The first definitions of the trigonometric functions works only for angles between 0 and $\pi/2$. To extend the definitions to work for all angles, we use a circle. Fix a circle of radius r and put a cartesian coordinate system with origin at the center of the circle.

Now lay off the angle θ with vertex at the origin and initial ray along the horizontal axis. The terminal ray intersects the circle at a point. Let (u, v) be the coordinates of this point. We then define the trigonometric functions by

$$\begin{aligned} \sin\theta &= \frac{u}{r} \\ \cos\theta &= \frac{v}{r} \\ \tan\theta &= \frac{u}{v} \end{aligned}$$



Exercise 6. Convince yourself that this new definition is the same as the first definition for angles between 0 and $\pi/2$ (i.e., angles in the first quadrant of the previous figure). Hint: Drop a perpendicular from the point (u, v) to the horizontal axis and look at the resulting right triangle. Think about what u and v mean in terms of this right triangle.

This new definition works for any value of θ. For example, with $\theta = 0$, the terminal ray also lies on the positive horizontal axis so the point of intersection has coordinates $(r, 0)$. Thus,

$$\sin 0 = \frac{0}{r} = 0 \qquad \text{and} \qquad \cos 0 = \frac{r}{r} = 1.$$

For $\theta = \pi/2$, the terminal ray lies on the positive vertical axis so the point of intersection has coordinates $(0, r)$. Thus,

$$\sin\tfrac{\pi}{2} = \frac{r}{r} = 1 \qquad \text{and} \qquad \cos\tfrac{\pi}{2} = \frac{0}{r} = 0.$$

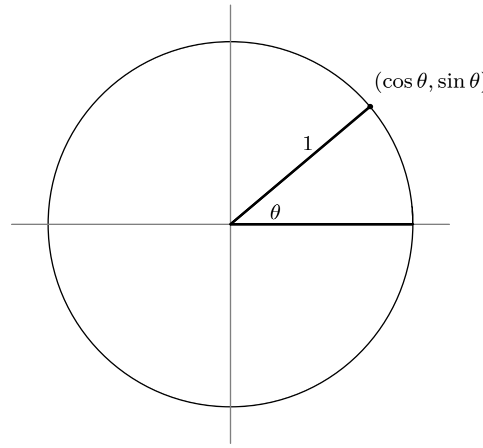
Exercise 7. Fill in the values in the table using your results from Exercise 3.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			

The unit circle

With the circle definition, values of the trigonometric functions depend on the angle and not on the radius of the circle we use. For simplicity, we can use a circle of radius 1, otherwise known as the *unit circle*. With this the definitions simplify to

$$\begin{aligned} \sin\theta &= u \\ \cos\theta &= v \\ \tan\theta &= \frac{u}{v} \end{aligned}$$

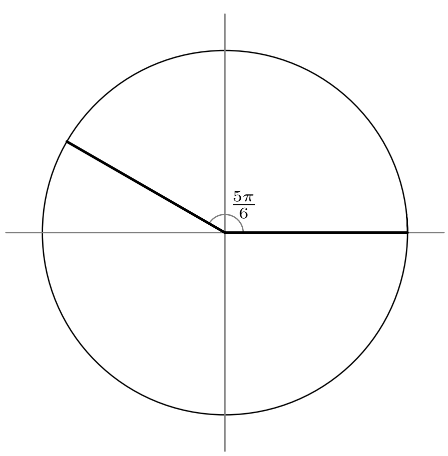


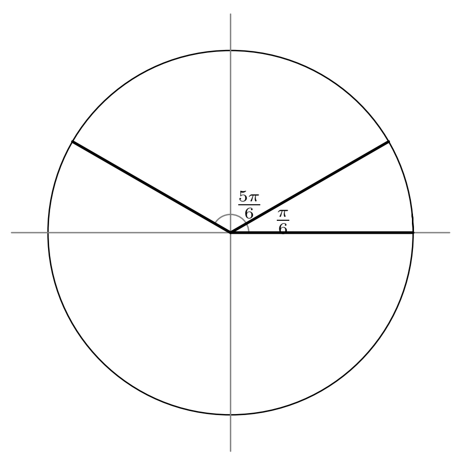
Notice that we can now think of the point on the circle as having coordinates $(\cos\theta, \sin\theta)$. That is, on the unit circle, the values of $\cos\theta$ and $\sin\theta$ are just the horizontal and vertical coordinates of this point.

Values for angles in Quadrants II, III, and IV.

We know the exact values of the trigonometric functions for the angles 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. We can use these values and symmetry to determine exact values of the trigonometric functions for other angles. For example, consider the angle $5\pi/6$ as shown in the figure below. Compare this with the angle $\pi/6$. By symmetry, the two corresponding points on the circle have opposite horizontal coordinates and equal vertical coordinates. Using the values from Exercise 7, we get

$$\cos\left(\tfrac{5\pi}{6}\right) = -\cos\left(\tfrac{\pi}{6}\right) = -\tfrac{1}{2} \qquad \text{and} \qquad \sin\left(\tfrac{5\pi}{6}\right) = \sin\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}.$$





In this example, the angle $\pi/6$ is called the *reference angle* for $5\pi/6$. Using the angles from Exercise 7 as reference angles, we can find the exact values of certain angles in Quadrants II, III, and IV.

Exercise 8. Fill in the values in the table on the left using reference angles, symmetry, and the values from Exercise 7. Ignore the table on the right for now.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			
π			
$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			
$\frac{3\pi}{2}$			
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			
2π			

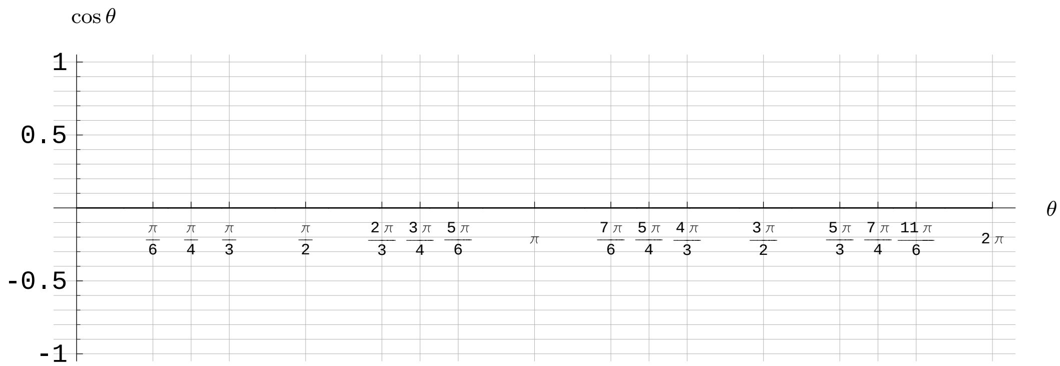
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			
π			
$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			
$\frac{3\pi}{2}$			
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			
2π			

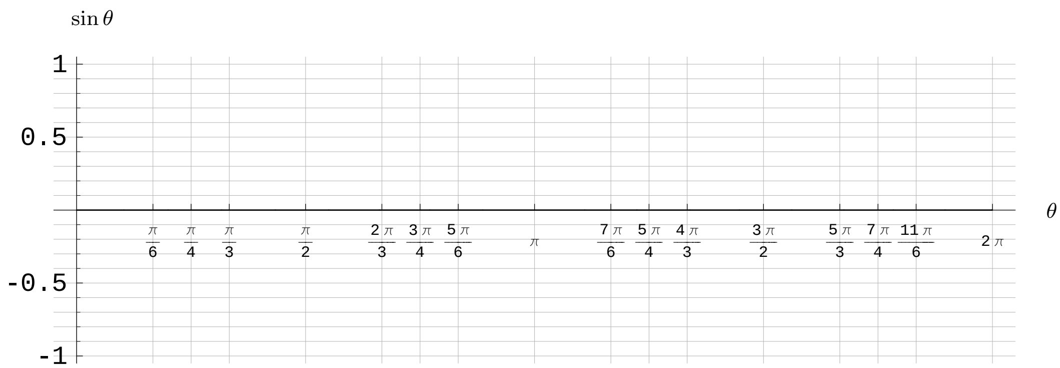
Graphs of the sine and cosine functions

In calculus, we will be emphasizing the *function* aspect of the trigonometric functions. For this it is useful to have in mind input-output graphs of these functions. The table in Exercise 8 has some input-output data we can plot. To plot points, it might be best to have decimal approximations of the output values.

Exercise 9. Fill in the table on the right above with decimal approximations rounded to the thousandth place.

Exercise 10. Use the data from Exercise 9 to carefully plot points on the axes that follow.





To fill in a continuous curve on these plots, we need to interpolate. From the definition of $\cos\theta$ and $\sin\theta$ as the horizontal and vertical coordinates of the point on the unit circle corresponding to the angle θ, we see that the range of these functions is $[-1, 1]$. By picturing the angle θ increasing from 0 to 2π and watching the coordinates of the points change, we see that the values of $\cos\theta$ and $\sin\theta$ vary continuously. For an example of this, go to the web site `http://clem.mscd.edu/ talmanl/MathAnim.html` and click on the link for the sine curve.

Exercise 11. Draw a continuous curve that interpolates the points you plotted in Exercise 10.