Antiderivatives and FTC in various settings

Theorem:
If $f : [a, b] \to \mathbb{R}$ is continuous, then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(u)\, du$$

is differentiable with $F'(x) = f(x)$.

Theorem:
If (i) G is an open convex region in $\mathbb{R}^2$

(ii) $\vec{F} : G \to \mathbb{R}^2$ is continous

(iii) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in G

then $V : G \to \mathbb{R}$ defined by

$$V(P) = \int_{P_0}^P \vec{F} \cdot d\vec{r}$$

(for a fixed point P_0 in G) is differentiable with $\vec{\nabla} V(P) = \vec{F}(P)$.

Theorem:
Let G be an open convex region in $\mathbb{R}^2$ and let $\vec{F} : G \to \mathbb{R}^2$ be continous. The following are equivalent:

1. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in G
2. $\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}$ for every pair of paths C_1 and C_2 in G that have the same initial point and have the same final point
3. There is a continuously differentiable function $V : G \to \mathbb{R}^2$ such that $\vec{\nabla} V(P) = \vec{F}(P)$ for all points P in G.

If, in addition, $\vec{F} : G \to \mathbb{R}^2$ is continuously differentiable, the following is also equivalent:

4. $\vec{\nabla} \times \vec{F}(P) = \vec{0}$ for all points in G