Name ______________________________

MATH 352 Complex Analysis Spring 2008 Final Exam

Instructions: Do your own work. You may consult your class notes and the course text. Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me.
Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning).
For problems with options, circle the label(s) of the part(s) you are submitting.
The exam is due at 10 am on Friday, May 16. (100 points total)

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1. A friend makes the following argument:

   Consider the function defined by
   $$f(z) = \cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$
   Note that
   $$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1 - 1/z} = -\frac{z}{1-z}.$$
   Also note that
   $$z + z^2 + z^3 + \cdots = z(1 + z + z^2 + \cdots) = z\frac{1}{1-z} = \frac{z}{1-z}.$$
   Therefore,
   $$f(z) = -\frac{z}{1-z} + \frac{z}{1-z} = 0.$$
   In other words, f is the zero function.

   Is your friend's argument correct? If not, explain any flaws in the argument. Is the conclusion correct? If f is not the zero function, determine what is true about f as a function. (12 points)

2. Do any three of the following four problems. (12 points each)

   (A) Prove the following: If f is holomorphic for a region G with $f(x+iy) = u(x,y) + iv(x,y)$ and $au(x,y) + bv(x,y) = c$ for real constants a, b, and c not all zero, then f is constant in A. Also, determine if the same conclusion holds if a, b, and c are complex constants.

   (B) Prove the following: If f is holomorphic for $\mathbb{C}$ and $\text{Im}(f(z)) \leq 0$ for all z, then f is a constant function.

   (C) Let f be holomorphic on an open set A. Define $B = \{z \in A | f(z) \neq 0\}$. Show that B is open and that $1/f$ is holomorphic on B.

   (D) Suppose f satisfies $|f(z)| < \dfrac{1}{1-|z|}$ on $D(0;1)$.

   (i) Get an upper bound on $|f^{(n)}(0)|$ by using the Cauchy Integral Formula for Derivatives with the relevant integrals computed along $\gamma(0;r)$. Your upper bound should depend on n and r.

   (ii) Find the value of r that minimizes the upper bound you get in (i) and thus get an optimal bound from (i). This will depend only on n.

3. Let C be the unit circle centered at the origin oriented counterclockwise. (7 points each)

   (a) Find the value of $\displaystyle\int_C \frac{\log z}{z}\,dz$ with the branch using $-\pi < \arg z \leq \pi$ for the logarithm.

   (b) Find the value of $\displaystyle\int_C \frac{\log z}{z}\,dz$ with the branch using $0 \leq \arg z < 2\pi$ for the logarithm.

4. Consider a function $f(z) = \dfrac{g(z)}{h(z)}$ where g and h are holomorphic at z_0, $g(z_0) \neq 0$, and h has a zero of order 2 at z_0.

   (a) Find an expression for the residue of f at z_0 in terms of g, h, and derivatives of these functions. (8 points)

   (b) Use your result to compute the residue of $f(z) = \dfrac{z+1}{\sin^2 z}$ at $z_0 = 0$. (6 points)

5. Do either one of the following two problems. (12 points)

   (A) Evaluate $\displaystyle\int_0^\infty \frac{1}{1+z^8}\,dz$ using complex techniques with each of the following contours.

      (i) the "standard" contour consisting of the segment $[-R, R]$ and the semicircle Γ_R.

      (ii) a more clever choice of contour that encloses only one relevant pole

   (B) Use complex techniques to find a formula for $\displaystyle\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}$ with $a, b > 0$.

6. Suppose the Math Club exists and it publishes a monthly newsletter for math students. Write an article that describes complex analysis for this newsletter. Consider your target audience to be math students who have completed the calculus sequence and linear algebra but who have *not* taken complex analysis. Your goal is to give those students guidance in making an informed decision about taking a course in complex analysis. Focus on aspects of complex analysis that you think are important or interesting. You do not need to summarize every idea that we have covered this semester. Your opinion of complex analysis does not need to be positive. I will evaluate your essay in terms of the following criteria: mathematical precision and accuracy; originality and insight; theme and coherence; technical aspects such as grammar and spelling. (12 points)