Name ______________________________

MATH 352 Complex Analysis Spring 2008 Exam #1

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (115 points total)

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1. Simplify each of the following. Give each result in the form $a + bi$ with a and b real. (4 points each)

   (a) $\overline{(2+i)^2}$ (b) $(-1+i)^7$ (c) $\dfrac{i}{-2+5i}$

2. Use a complex analysis approach to find an identity for $\cos(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$. (8 points)

3. Find all solutions of the equation $1 + 3z + 9z^2 + 27z^3 = 0$. You can give the solutions in polar form or in cartesian form. (8 points)

4. Let w_1, w_2, and w_3 be the cube roots of unity. Make a sketch to demonstrate the fact that $w_1 + w_2 + w_3 = 0$. (6 points)

5. For each of the following, sketch the given set and categorize it using ideas we've discussed (such as open, not open, closed, bounded, etc.). (7 points each)

   (a) $1 < |z + \sqrt{3} + i| < 2$ (b) $|z| = \arg z$ (c) $\dfrac{|z+1|}{|z-i|} \geq 2$

6. Do any two of the following four problems. Circle the problem numbers for the two problems you are submitting. (6 points each)

   (a) Prove the following: If $|\bar{z} - 1| \neq 0$, then $\left|\dfrac{1-z}{\bar{z}-1}\right| = 1$.

   (b) Prove the following: If $|z| \leq 1$, then $|\operatorname{Re}(2 + \bar{z} + z^3)| \leq 4$.

   (c) Prove the following: If $p(z)$ is polynomial with real coefficients and $p(\alpha) = 0$, then $p(\bar{\alpha}) = 0$.

   (d) Prove the following: For any n, the nth roots of unity sum to 0. Hint: Use the fact that the nth roots of unity can be written as $1, \omega_0, \omega_0^2, \ldots, \omega_0^{n-1}$ for $\omega_0 \neq 1$.

There is more on the flip side.

7. (a) Find the Mobius transformation that maps $-i$, 0, i to -1, i, 1, respectively. (6 points)

   (b) For the Mobius transformation from (a), find the image of the imaginary axis. (3 points)

   (c) For the Mobius transformation from (a), find the image of the real axis. (3 points)

8. For each of the following, determine whether or not the given sequence converges. Give a proof to support each conclusion. (6 points each)

   (a) $z_n = \dfrac{3n+i}{4+7ni}$ (b) $z_n = \dfrac{3n^2+i}{4+7ni}$

9. For each of the following, determine the set on which the given function is continuous. Give some justification to support each conclusion. (6 points each)

   (a) $f(z) = \dfrac{z+3}{z^3+z}$ (b) $f(z) = \begin{cases} \dfrac{z+i}{z^2+1} & \text{for } z \neq -i \\ \frac{1}{2}i & \text{for } z = -i \end{cases}$

10. Do any two of the following four problems. Circle the problem numbers for the two problems you are submitting. (6 points each)

   (a) Prove the limit statement $\displaystyle\lim_{z \to 2i} \frac{z^2+4}{5z-10i} = \frac{4}{5}i$.

   (b) Prove the following: If $z_n \to z$, then $\bar{z}_n \to \bar{z}$.

   (c) Prove the following: If $z_n \to z$ and α is any complex number, then $\alpha z_n \to \alpha z$.

   (d) Prove the following: If $\displaystyle\lim_{z \to z_0} f(z) = \alpha$, then $\displaystyle\lim_{z \to z_0} |f(z)| = |\alpha|$.