Choosing a branch for a multifunction involving cube root

Note:I will use a single square bracket to denote a multifunction rather than the text's "double square bracket". Also, I haven't proofread this carefully. If you spot mistakes or want clarification, send me a note or stop by.

To get started, let's look at branches of $[z^{1/3}]$. Choose θ in $[\arg z]$ so we can write

$$\begin{aligned}
[z^{1/3}] &= \left\{ e^{\frac{1}{3}(\ln|z|+i(\theta+2\pi k))} \middle| k \in \mathbb{Z} \right\} \\
&= \left\{ e^{\frac{1}{3}(\ln|z|+i(\theta+2\pi k))} \middle| k = 0,1,2 \right\} \\
&= \left\{ e^{\frac{1}{3}\ln|z|+i\frac{\theta}{3}}, e^{\frac{1}{3}\ln|z|+i\frac{\theta}{3}} e^{i\frac{2\pi}{3}}, e^{\frac{1}{3}\ln|z|+i\frac{\theta}{3}} e^{i\frac{4\pi}{3}} \right\} \\
&= \left\{ F_0(z), F_1(z), F_2(z) \right\}
\end{aligned}$$

Note that

$$F_0(z) = e^{i\frac{2\pi}{3}} F_2(z), \qquad F_1(z) = e^{i\frac{2\pi}{3}} F_0(z), \qquad \text{and} \qquad F_2(z) = e^{i\frac{2\pi}{3}} F_1(z).$$

Now consider a circuit around a circle centered at 0. Start at any point z_0 on the circle and pick a particular branch, say F_0. As z traverses the circle once, it picks up a factor of $e^{i2\pi}$ so $F_0(z)$ picks up a factor of $e^{i2\pi/3}$ to end with value $e^{i2\pi/3}F_0(z_0) = F_1(z_0)$. This circuit takes us from one branch to another. Given the relations above, continuing to traverse this circuit will give us the pattern $F_0 \to F_1 \to F_2 \to F_0$.

Let's apply these ideas to find a branch for $[(z^2-1)^{1/3}]$. We can write $(z^2-1)^{1/3}$ in factored form as $(z-1)^{1/3}(z+1)^{1/3}$. The nature of the factor $(z-1)^{1/3}$ at 1 is identical to the nature of $z^{1/3}$ at 0. In considering a circuit around a small circle centered at 1, the presence of the other factor, $(z+1)^{1/3}$, does not change the conclusion that output values change from one branch to another since $\arg(z+1)$ only varies in a small range (and certainly not through a range of 2π). In similar fashion, we can argue that the nature of the factor $(z+1)^{1/3}$ at -1 identical to the nature of $z^{1/3}$ at 0. So, we conclude that 1 and -1 are branch points of $[(z^2-1)^{1/3}]$.

To choose a suitable branch cut, we need to analyze other circuits. The only other interesting case is a circuit encircling both branch points. For such a circuit, $\arg(z+1)$ and $\arg(z-1)$ both increase by 2π so $\arg(z+1)+\arg(z-1)$ increases by 4π. This leads to an additional factor of $e^{i4\pi/3}$ in the value of any branch of $[(z^2-1)^{1/3}]$ we choose to start on. The presence of this factor (which is not equal to 1) tells us that we end the circuit on a different branch than the one we start on. So, circuits of this type are inadmissible and we must choose branch cut(s) to preclude these.

One choice is to use the branch cuts $(-\infty,-1]$ and $[1,\infty)$ on the real axis. With this choice, we have a domain on which we can define a single-valued function $f(z)$ that is a branch of $[(z^2-1)^{1/3}]$. To make a specific choice of branch, we can specify a value at one point. For example, we could choose one value from $[(0^2-1)^{1/3}] = [(-1)^{1/3}] = \{e^{i\pi/3}, e^{i\pi} = 1, e^{i4\pi/3}\}$ as $f(0)$. Each of the three possible choices gives a different branch, with all three branches having the same domain, namely $\mathbb{C}\backslash(-\infty,-1]\cup[1,\infty)$.

For completeness, let's check if $[w(z)] = [(z^2-1)^{1/3}]$ has a branch point at infinity. To do this, we check if $[w(1/z)]$ has a branch point at 0. So, compute

$$w\left(\frac{1}{z}\right) = \left(\frac{1}{z^2} - 1\right)^{1/3} = \left(\frac{1-z^2}{z^2}\right)^{1/3} = \frac{(1-z^2)^{1/3}}{z^{2/3}}.$$

Because of the factor $z^{2/3}$ in the denominator, this does have a branch point at 0. (Note that this also has branch points at -1 and 1 but these are not relevant here.) The presence of a branch point at infinity for $[(z^2-1)^{1/3}]$ tells us that we must have at least one branch cut that extends to ∞.

You should do the calculation to confirm that $[(z^2-1)^{1/2}]$ does not have a branch point at infinity. For this multifunction, it is not necessary to use a branch cut that extends to ∞ (although it is allowed to use one).