Versions of Cauchy's Theorem

We have proven the following (given in 11.2 on page 129 of Priestley):

Theorem 1A: If f is holomorphic on and inside a triangular path γ, then $\int_\gamma f(z)\,dz = 0$.

In 11.6, Priestley generalizes Theorem 1A to the following:

Theorem 1B: If f is holomorphic on and inside a closed contour γ, then $\int_\gamma f(z)\,dz = 0$.

Recall that Priestley defines a *contour* as a simple path that is the join of line segments and circular arcs. I think Priestley restricts attention to the special case of a contour (in his sense) since he can provide a complete proof on his "basic track." In particular, Priestley provides a proof of the Jordan Curve Theorem (needed to give meaning to "inside") for the restricted case of closed contours. A proof of the Jordan Curve Theorem for simple closed paths is much more difficult.

Without giving a detailed proof, we will accept the following generalization of Theorem 1B:

Theorem 1C: If f is holomorphic on and inside a simple closed path γ, then $\int_\gamma f(z)\,dz = 0$.

In Chapter 12 (the "advanced track"), Priestley presents versions of Cauchy's theorem that are still more general. These versions have a slightly different formulation. Recall that a *region* is a nonempty, open, connected subset of $\mathbb{C}$. The first version of this type actually appears in Chapter 11 because it is used in the proof of Theorem 1B.

Theorem 2A: If f is holomorphic in a convex region G and γ is a closed path in G, then $\int_\gamma f(z)\,dz = 0$.

The version given in 12.5 uses the notion of *simply connected*. Here's an intuitive definition: *A region G is simply connected if the inside of every simple closed path in G contains only points in G.* Priestley gives a more technical definition that has the advantage of being easier to apply.

Here's the version of Cauchy's Theorem given in 12.5:

Theorem 2B: If f is holomorphic in a simply connected region G and γ is a closed path in G, then $\int_\gamma f(z)\,dz = 0$.

Priestley's final version of Cauchy's Theorem, given in 12.12, replaces the condition that the region G be simply connnected with a condition on the closed curve γ. This is condition on γ is phrased in terms of a quantity called the *index* of γ with respect to a point w not on γ^*. On the intuitive level, the index of a closed path γ with respect to a point w is a nonnegative integer that tells us how many times "γ winds around w". The index is denoted $n(\gamma, w)$. Here's Priestley's final version of Cauchy's Theorem:

Theorem 2C: If f is holomorphic in a region G and γ is a closed path in G such that $n(\gamma, w) = 0$ for all $w \notin G$, then $\int_\gamma f(z)\,dz = 0$.