Two ways to CIF-2 and beyond

1. Just do the algebra:

   (a) Use CIF-1 to re-express $\dfrac{f'(z+h)-f'(z)}{h}$.

   (b) Simplify your result from (a) using straightforward algebra. Be clever in the way you do the algebra.

   (c) Substitute your result from (b) into $\left|\dfrac{f'(z+h)-f'(z)}{h} - \dfrac{2}{2\pi i}\displaystyle\int_\gamma \dfrac{f(w)}{(w-z)^3}\,dw\right|$.

   (d) Simplify your result from (c) using straightforward algebra. Be clever in the way you do the algebra. Your goal is to get an expression in the form of the modulus of a single integral, so $\left|\int_\gamma A\,dw\right|$ for some expression A.

   (e) Use the Estimation Theorem to get an upper bound for your expression in (d) that is valid for all h such that $|h| < r/2$.

2. Avoid the algebra by being clever:

   (a) Use CIF-1 to re-express $\dfrac{f'(z+h)-f'(z)}{h}$.

   (b) Simplify your result from (a) by being clever.

   (c) Substitute your result from (b) into $\left|\dfrac{f'(z+h)-f'(z)}{h} - \dfrac{2}{2\pi i}\displaystyle\int_\gamma \dfrac{f(w)}{(w-z)^3}\,dw\right|$.

   (d) Simplify your result from (c) using the same clever idea you came up with in (b). Your goal is to get an expression in the form of the modulus of a single integral, so $\left|\int_\gamma A\,dw\right|$ for some expression A.

   (e) Use the Estimation Theorem to get an upper bound for your expression in (d) that is valid for all h such that $|h| < r/2$. Hint: To help in getting bounds on everything, draw a picture showing the geometric relation of all the relevant quantities.

3. Extending to CIF-n by induction:

   (a) Convince yourself that we've established the base case.

   (b) Set up the induction hypothesis that CIF-k holds.

   (c) Show that CIF-$(k+1)$ holds using the pattern from 2.