Error in Simpson's rule

We will build an approximation for a definite integral based on the interpolating polynomial on three points $\{x_0, x_1, x_2\}$. We start with

$$f(x) = p_2(x) + E_2(x)$$

and integrate both sides to get

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} p_2(x)\,dx + \int_{x_0}^{x_2} E_2(x)\,dx.$$

You can work out the details to find

$$\int_{x_0}^{x_2} p_2(x)\,dx = \frac{h}{2}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big)$$

which is our approximation.

The error in this approximation is given by

$$\int_{x_0}^{x_2} E_2(x)\,dx = \int_{x_0}^{x_2} \frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2)\,dx.$$

We cannot apply the Weighted Mean Value Theorem directly since $(x - x_0)(x - x_1)(x - x_2)$ changes sign on $[x_0, x_2]$. Following Andy's suggestion, we can split the interval at x_1 to get

$$\int_{x_0}^{x_2} E_2(x)\,dx = \int_{x_0}^{x_1} \frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)\,dx + \int_{x_1}^{x_2} \frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)\,dx.$$

Applying the Weighted Mean Value Theorem to each term, we conclude there exists xi_0 in $[x_0, x_1]$ and ξ_1 in $[x_1, x_2]$ such that

$$\int_{x_0}^{x_2} E_2(x)\,dx = \frac{f'''(\xi_0)}{3!}\int_{x_0}^{x_1} (x-x_0)(x-x_1)(x-x_2)\,dx + \frac{f'''(\xi_1)}{3!}\int_{x_1}^{x_2} (x-x_0)(x-x_1)(x-x_2)\,dx.$$

The remaining integrals are straightforward to evaluate. (Full disclosure: I coaxed them out of *Mathematica*.) After some simplification, we get

$$\int_{x_0}^{x_2} E_2(x)\,dx = \frac{f'''(\xi_0)}{3!}\Big(\frac{h^4}{4}\Big) + \frac{f'''(\xi_1)}{3!}\Big(-\frac{h^4}{4}\Big) = \frac{h^4}{24}\Big(f'''(\xi_0) - f'''(\xi_1)\Big).$$

Putting the pieces together, we have

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{2}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big) + \frac{h^4}{24}\Big(f'''(\xi_0) - f'''(\xi_1)\Big).$$

Having worked out the details of the error term, we can see that Simpson's rule is (unexpectedly) exact for cubic functions since the third derivative of a cubic function is constant.