Name ________________________________

MATH 280 Multivariate Calculus Fall 2006 Final Exam

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem. (100 points total)

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1. (a) Give a component proof for the identity $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$. (5 points)

   (b) Give a geometric argument for the identity $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$. (5 points)

2. The position of a fly buzzing around the room is given by $\vec{R}(t) = -3t^2\,\hat{\imath} + (1 - 3t)\,\hat{\jmath} + t^3\,\hat{k}$.

   (a) Compute the velocity and acceleration functions for the fly. (6 points)

   (b) Give the unit vector in the direction the fly is moving at $t = 2$. (4 points)

   (c) Find a time t for which the velocity of the fly is perpendicular to the acceleration of the fly or show that no such time exists. (4 points)

3. Consider the helix parametrized by $x = 2\cos t$, $y = 2\sin t$, and $z = 5t$. Find the equation of the line tangent to the helix at the point $(0, 2, 10\pi)$. (8 points)

4. Consider the two curves given by the vector-output functions $\vec{R}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} + t^3\,\hat{k}$ and $\vec{P}(s) = (s + 3)\,\hat{\imath} + (2s^2 + 2)\,\hat{\jmath} + (9 - s^4)\,\hat{k}$.

   (a) Confirm that each curve contains the point $(2, 4, 8)$. (4 points)

   (b) Find the angle between the curves at the point of intersection $(2, 4, 8)$. (8 points)

5. Compute all first and second partial derivatives of the function $f(x, y, z) = xz\,e^{yz}$. (12 points)

6. Find the equation of the tangent plane for $f(x, y) = x^3 y$ for $(2, 5)$. (12 points)

7. Show that $(3, -1)$ is a critical input for the function $f(x, y) = x^2y^3 - 6xy - 9y$ and classify this input as a local minimizer, a local maximizer, or neither. (12 points)

8. Motion of air in the atmosphere (i.e., wind) is related to differences in atmospheric pressure from one place to another. In a simple-minded way of thinking about this, air moves in the direction atmospheric pressure decreases most rapidly. With this thinking, find the direction air moves at the point $(2, 4, 1)$ if the atmospheric pressure is given by $p(x, y, z) = 4xy + 3yz$ (in unspecified units). (10 points)

9. Charge is distributed in a solid sphere of radius R. The charge density (per unit volume) is $\delta = kr\sin(\theta/2)$ where k is a constant and (ρ, ϕ, θ) are spherical coordinates with origin at the center of the sphere. Compute the total charge for the sphere. (12 points)

10. Set up and evaluate an interated integral (or integrals) to compute the value of the double integral of the function $f(x, y) = x^2 y$ over the region R in the xy-plane bounded below by $y = x^3$ and above by $y = \sqrt{32x}$. (20 points)

11. Set up, but do not evaluate, an iterated integral (or integrals) that gives the value of the triple integral of the function $f(x, y, z) = xy^2 + z^2$ over the solid region bounded by the planes $x = 0$, $y = 0$, $y = 5$, $z = 0$, and $z = 4 - 2x$. (20 points)

12. Set up, but do not evaluate, an iterated integral (or integrals) that gives the volume of the solid region that is above the xy-plane and inside both the sphere of radius 10 centered at the origin and the cone $z = \sqrt{x^2 + y^2}$. Express your result entirely in terms of a single coordinate system (of your choice). (20 points)

13. Set up, but do not evaluate, an iterated integral (or integrals) that gives the total mass of the top half of the solid sphere of radius R centered at the origin with a mass density proportional to the square of the height above the xy-plane. Express your result entirely in terms of a single coordinate system (of your choice). (20 points)

14. Compute the formula for the volume of a sphere of radius R. (12 points)

15. Consider the vector field $\vec{F}(x, y) = y^2\,\hat{\imath}$.

    (a) Sketch a plot of this vector field in the xy-plane. (4 points)

    (b) Use your vector field plot to determine geometrically where in the xy-plane the divergence is positive, where the divergence is negative, and where the divergence is zero. Make your reasoning clear. (3 points)

    (c) Compute the divergence of $\vec{F}(x, y)$. Compare this to your results for (b). (4 points)

    (d) Use your vector field plot to determine geometrically where in the xy-plane the $\hat{k}$-component of the curl is positive, where it is negative, and where the it is zero. Make your reasoning clear. (3 points)

    (e) Compute the curl of $\vec{F}(x, y)$. Compare this to your results for (d). (4 points)

16. Consider the line integral $\int_C \vec{F} \cdot d\vec{R}$ where $\vec{F}(x, y, z) = z\,\hat{\imath} + z\,\hat{\jmath} + (x + y)\,\hat{k}$ and C is the straight line from $(2, 1, 3)$ to $(5, 0, 1)$.

    (a) Compute $\displaystyle\int_C \vec{F} \cdot d\vec{R}$ by parametrizing the curve. (8 points)

    (b) Compute $\displaystyle\int_C \vec{F} \cdot d\vec{R}$ using the Fundamental Theorem for Line Integrals. (8 points)