Name ______________________________

MATH 221B | Multivariate Calculus | Spring 2004 | Final Exam

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

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1. Consider the functions

$$f(x,y) = x^2y + y^3 \qquad \text{and} \qquad \vec{F}(t) = (t-4)\,\hat{\imath} + \sqrt{t+1}\,\hat{\jmath} + (t^2+1)\,\hat{k}.$$

   (a) Show that the graph of f and the image curve C for $\vec{F}$ intersect at the point $(-1, 2, 10)$. (4 points)

   (b) Find the equation of the tangent plane for f at the point $(-1, 2, 10)$. (5 points)

   (c) Compute $\vec{F}'$ for the point $(-1, 2, 10)$. (4 points)

   (d) Find the acute angle between a normal vector for the graph of f and a tangent vector for the curve C at the point $(-1, 2, 10)$. (5 points)

2. Find and classify (as local minimizer, local maximizer, or neither) all critical points of the function $f(x,y) = x^3 + y^2 - 6xy + 24x + 7$. (12 points)

3. Suppose the temperature in a particular room is given by $T = xy\cos(\pi z)$ where T is measured in $^\circ C$ and x, y, and z are measured in meters.

   (a) Compute the gradient $\vec{\nabla} T$ for the point $(2, 3, 4)$. (4 points)

   (b) Find the rate of change in T with respect to change in position at the point $(2, 3, 4)$ in the direction of the origin. (5 points)

   (c) Find one direction (a unit vector) tangent to the level surface of T at the point $(2, 3, 4)$. (3 points)

4. Set up, but do not evaluate, an iterated integral (or integrals) to give the volume of the solid region inside both the cylinder $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$. (12 points)

5. Consider a solid sphere of radius R. In spherical coordinates (ρ, ϕ, θ) with origin at the center of the sphere, the mass density is given by $\delta = k\rho^3$ where k is a constant with units of kg/m^6. Find the total mass of the sphere. (12 points)

6. Consider the vector field $\vec{F} = ax^2\,\hat{\imath} + (y - a^2 z)\,\hat{\jmath} + (ay + z^2)\,\hat{k}$ where a is a constant.

   (a) Find the values of a for which the divergence of $\vec{F}$ at the point $(3, 5, 1)$ is positive. (6 points)

   (b) Find the values of a for which the vector field $\vec{F}$ is conservative. (6 points)

7. Consider the vector field $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2 + y^2}$. Let C be the circle of radius 3 centered at the origin with clockwise orientation.

   (a) Plot the vector field $\vec{F}$. Show enough detail to give a general sense of the vector field. (4 points)

   (b) Draw C on your vector field plot. Determine the sign of $\oint_C \vec{F} \cdot d\vec{R}$ from this plot. Briefly explain how you reach your conclusion. (4 points)

   (c) Compute the value of $\oint_C \vec{F} \cdot d\vec{R}$. (6 points)

8. Suppose $\vec{F}$ is a vector field with $\vec{\nabla} \cdot \vec{F} = 0$ for all points in $\mathbb{R}^3$. show that $\oiint_S \vec{F} \cdot d\vec{A} = 0$ for any closed surface S in $\mathbb{R}^3$. (8 points)