Line integral example

Problem Charge is distributed on a semicircle of radius R. The length charge density is proportional to the distance from the diameter that contains the two ends of the semicircle. Let λ_0 be the maximum charge density. Compute the total charge on the semicircle.
The total charge Q is given by the line integral

$$Q = \int_{\text{semicircle}} \lambda \, ds$$

where λ is the length charge density and $ds = |d\vec{r}|$ is the line element. We'll compute this quantity with three different methods for comparison. In each method, the goal is to express the line integral as an definite integral with respect to a single variable.

Method 1: Cartesian coordinates

We choose cartesian coordinate axes with origin at the center of the circle and x-axis along the diameter containing the ends of the semicircle.
Coordinate expression for the integrand:
In this coordinate system, y gives the distance from the diameter containing the ends of the semicircle. The length charge density is thus proportional to y so $\lambda = ky$ for some proportionality constant k. The maximum charge density will occur for $y = R$ so $\lambda_0 = kR$ which we solve to get $k = \lambda_0/R$. We thus have

$$\lambda = \frac{\lambda_0}{R}\, y.$$

Coordinate expression for the length element:
The equation of the circle is $x^2 + y^2 = R^2$. We can "d" this to get $2x\,dx + 2y\,dy = 0$. Since the coordinate expression λ involves y, we will set up our definite integral in terms of y with y ranging from 0 to R. We do need to be a bit careful with this since each small interval of size dy along the y-axis corresponds to two pieces of the curve. We'll take care of this with a factor of 2 when we put everything together.

Having picked y, we need to express the length element ds in terms of y and dy. To do this, we will first eliminate dx by solving to get

$$dx = -\frac{y}{x}\, dy$$

Substituting from this, we have

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} = -\frac{y}{x}\, dy\,\hat{\imath} + dy\,\hat{\jmath}$$

and thus

$$ds = |d\vec{r}| = \sqrt{\left(-\frac{y}{x}\, dy\right)^2 + \left(dy\right)^2} = \sqrt{\frac{y^2}{x^2} + 1}\;|dy|.$$

Note that we have $|dy|$ since in general $\sqrt{a^2} = |a|$. We still need to substitute for x^2 using the equation of the circle to get $x^2 = R^2 - y^2$. Making this substitution and doing some algebra, we get

$$ds = \frac{R}{\sqrt{R^2 - y^2}}\,|dy|.$$

Putting it all together:
Using the coordinate expressions for the integrand and the line element, we have

$$Q = \int_{\text{semicircle}} \lambda\, ds = 2\int_0^R \frac{\lambda_0}{R}\,y\frac{R}{\sqrt{R^2 - y^2}}\,|dy| = 2\lambda_0 \int_0^R \frac{y}{\sqrt{R^2 - y^2}}\,dy.$$

The factor of 2 on the right side is the one mentioned above to account for the fact that each dy corresponds to two pieces of the semicircle. Note that we can use $|dy| = dy$ since y increases from 0 to R.

We have now expressed the line integral as a definite integral in one variable. After evaluating this definite integral (as we did in class), we get

$$Q = 2\lambda_0 \int_0^R \frac{y}{\sqrt{R^2 - y^2}}\,dy = 2\lambda_0 R.$$

Method 2: Polar coordinates

We choose polar coordinate axes with origin at the center of the circle reference ray running along the diameter containing the ends of the semicircle.
Coordinate expression for the integrand:
From what we did above, we have

$$\lambda = \frac{\lambda_0}{R}\,y.$$

The cartesian coordinate y is related to polar coordinates by $y = r\sin\theta$. Along the semicircle, we have $r = R$ so we have

$$\lambda = \frac{\lambda_0}{R}\,R\sin\theta = \lambda_0\sin\theta.$$

Coordinate expression for the length element:
It is natural to use θ as the variable of integration with θ ranging from 0 to π. Since the curve here is a semicircle, we know from geometry that a small change of $d\theta$ corresponds to a small displacement of $ds = R\,d\theta$.
Putting it all together:
Using the coordinate expressions for the integrand and the line element, we have

$$Q = \int_{\text{semicircle}} \lambda\, ds = \int_0^\pi \lambda_0 \sin\theta\, R\,d\theta = \lambda_0 R\int_0^\pi \sin\theta\, R\,d\theta$$

It is straightforward to evaluate the definite integral in θ to get

$$Q = 2\lambda_0 R.$$

Method 3: Parametrizing the curve

Finding a parametrization: From previous experience, we know that the semicircle is traced out by the vector-output function

$$\vec{r}(t) = R\cos t\,\hat{\imath} + R\sin t\,\hat{\jmath}$$

for t ranging from 0 to π. In terms of components, this is equivalent to $x = R\cos t$ and $y = R\sin t$.

Parameter expression for the integrand:

From what we did above, we have

$$\lambda = \frac{\lambda_0}{R}\,y.$$

The cartesian coordinate y is given by $y = R\sin t$ so we have

$$\lambda = \frac{\lambda_0}{R}\,R\sin t = \lambda_0 \sin t.$$

Parameter expression for the length element:

We can "d" the vector-output function $\vec{r}(t)$ to get

$$d\vec{r} = -R\sin t\,dt\,\hat{\imath} + R\cos t\,dt\,\hat{\jmath} = \left(-R\sin t\,\hat{\imath} + R\cos t\,\hat{\jmath}\right)dt.$$

Computing the magnitude, we get

$$ds = |d\vec{r}| = \sqrt{\left(-R\sin t\right)^2 + \left(R\cos t\right)^2 (dt)^2} = R\sqrt{\sin^2 t + \cos^2 t}\;|dt| = R\,|dt|$$

Putting it all together:

Using the parameter expressions for the integrand and the line element, we have

$$Q = \int_{\text{semicircle}} \lambda\;ds = \int_0^{\pi} \lambda_0 \sin t\,R\,dt = \lambda_0 R \int_0^{\pi} \sin t\,R\,dt$$

It is straightforward to evaluate the definite integral in t to get

$$Q = 2\lambda_0 R.$$

Notes:

1. The details of Methods 2 and 3 are very similar in this case because the parameter t is really just the angle θ. Note that we got $ds = R\,d\theta$ in Method 2 from geometry while we got $ds = R\,dt$ in Method 3 by a computation.

2. Note that we saved a bit of reasoning in Methods 2 and 3 by using some reasoning from Method 1 in thinking about the integrand. You should take this into account in making a fair comparison of which method is best for you.

3. In many ways, Method 2 is the most natural. Method 1 is based on the familiar cartesian equation $x^2 + y^2 = R^2$ for the circle but leads to messier algebra and a more difficult definite integral to evaluate.

4. Method 3 is essentially automatic after you have written down a valid parametrization of the curve. On the other hand, coming up with a valid parametrization is not always obvious.