Name ______________________________

MATH 352 Complex Analysis Spring 2006 Final Exam

Instructions: Use separate paper for your responses. Please use just one side of each sheet and clearly number the problems. You can work on the problems in any order. Each statement of an axiom, definition, or theorem should be equivalent to that in the text.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

Note: All simple closed curves have positive orientation unless stated otherwise.

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1. Simplify $\dfrac{3-4i}{2-3i} + 5i$ to the form $a + bi$ where a and b are real numbers. (6 points)

2. Show that the exponential function $f(z) = e^z$ is entire and that $f'(z) = e^z$. (8 points)

3. Prove the identity $\sin^2 z + \cos^2 z = 1$. (6 points)

4. Find a series expansion for $f(z) = \dfrac{1}{z^2(1+z^2)}$ that is valid in the region $1 < |z|$. (8 points)

5. For each of the following, compute the value of the given contour integral. Give each result in the form $a + bi$ where a and b are real. (8 points each)

   (a) $\displaystyle\oint_C \frac{e^z}{z^3 - z}\,dz$ where C is circle of radius 1 centered at $z = \frac{1}{2}$

   (b) $\displaystyle\int_C z^i\,dz$ where C is a contour from 1 to -1 that lies above the real axis (except for the endpoints)

   (c) $\displaystyle\oint_C e^z \cos z\,dz$ where C is the unit circle centered at the origin

6. Use techniques from complex analysis to evaluate the improper integral $\displaystyle\int_0^\infty \frac{x^2}{x^4+1}\,dx$. (9 points)

There is more on the flip side.

7. Do any three of the following five problems. Circle the problem numbers for the problem you are submitting. (8 points each)

(A) Use the definition of limit to prove the following statement: If $\lim_{z \to z_0} f(z) = \alpha$ and $\lim_{z \to z_0} g(z) = \beta$, then $\lim_{z \to z_0} [f(z) + g(z)] = \alpha + \beta$.

(B) Show that $\operatorname{Log} z$ defined for $-\pi < \arg z \leq \pi$ is *not* continuous for any point on the negative imaginary axis.

(C) Show that the function $f(z) = |z|^2$ is differentiable but not analytic for $z = 0$.

(D) Prove the *curve deformation theorem*: Let C_1 and C_2 be positively oriented simple closed curves with C_2 inside C_1. If f is analytic for all points on and between these curves, then $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

(E) Prove the following: Suppose p and q are analytic at z_0 with $p(z_0) \neq 0$ and $q(z_0) = 0$. If $\dfrac{p(z)}{q(z)}$ has a pole of order m at z_0, then q has a zero of order m at z_0.