Name ______________________________

MATH 352 Complex Analysis Spring 2006 Exam #4

Instructions: Do your own work. You may consult your class notes and the course text. Do not consult other sources. Do not discuss generalities or specifics of the exam with anyone except me.

Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning). If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong or on alternate approaches you might try.

Each problem has a maximum value of 20 points.

The exam is due Wednesday, May 3 at 8:00 am.

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1. For each of the following regions, find the Laurent series expansion for the function $f(z) = \dfrac{1}{1+z^2} + \dfrac{1}{3-z}$ that is valid for that region.

   (a) $|z| < 1$ (b) $1 < |z| < 3$ (c) $3 < |z|$

2. A friend makes the following argument:

   Consider the function defined by

   $$f(z) = \cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

   Note that

   $$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1 - 1/z} = -\frac{z}{1-z}.$$

   Also note that

   $$z + z^2 + z^3 + \cdots = z(1 + z + z^2 + \cdots) = z\frac{1}{1-z} = \frac{z}{1-z}.$$

   Therefore,

   $$f(z) = -\frac{z}{1-z} + \frac{z}{1-z} = 0.$$

   In other words, f is the zero function.

   Is your friend's argument correct? If not, explain any flaws in the argument. Is the conclusion correct? If f is not the zero function, determine what is true about f as a function.

3. Consider a function $f(z) = \dfrac{p(z)}{q(z)}$ where p and q are analytic at z_0, $p(z_0) \neq 0$, and q has a zero of order 2 at z_0.

   (a) Find an expression for the residue of f at z_0 in terms of p, q, and derivatives of these functions.

   (b) Use your result to compute the residue of $f(z) = \dfrac{z+1}{\sin^2 z}$ at $z_0 = 0$.

There is more on the flip side.

4. Evaluate the contour integral $\oint_C \frac{z^4}{(z+1)(z-2)(z-2i)}\,dz$ where C is the circle of radius 3 centered at the origin.

5. Do one of the following two problems.

   (a) Problem 8 on p. 257 of the text.

   (b) Find the value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by analyzing $\oint_{C_N} \frac{1}{z^2 \tan z}\,dz$ where C_N is the square contour with edges on the lines $x = \pm\left(N + \frac{1}{2}\right)\pi$ and $y = \pm\left(N + \frac{1}{2}\right)\pi$.