

**Instructions:** Use separate paper for your responses. Please use just one side of each sheet and clearly number the problems. You can work on the problems in any order. Each statement of an axiom, definition, or theorem should be equivalent to that in the text.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

Note: All simple closed curves have positive orientation unless stated otherwise.

1. For each of the following, give a statement equivalent to that in text. Include hypotheses and conclusion for each. (7 points each)

- (a) Cauchy-Goursat Theorem
- (b) Cauchy Integral Formula for Derivatives
- (c) Liouville's Theorem

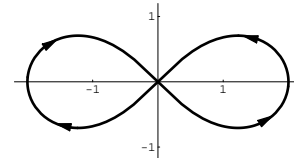
2. For each of the following, compute the value of the given contour integral. (9 points each)

(a)  $\oint_C e^{1/z} dz$  where  $C$  is the circle of radius 2 centered at  $3 + 5i$

(b)  $\oint_C \frac{z}{z^2 - 2z + 2} dz$  where  $C$  is the rectangle with vertices at 0, 2,  $2 + 2i$ , and  $2i$

(c)  $\oint_C \frac{\sin z}{z^2} dz$  where  $C$  is the unit circle centered at the origin

(d)  $\oint_C \frac{1}{z^2 - 1} dz$  where  $C$  is as shown here



3. Let  $C_R$  be the circle of radius  $R$  centered at the origin with  $R > 1$ . Find an upper bound on

$$\left| \oint_{C_R} \frac{z-1}{z(z^2+1)} dz \right|.$$

The upper bound can depend on  $R$ .

(9 points)

4. Briefly describe two essential and interesting ways in which differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  differ from analytic functions  $f : \mathbb{C} \rightarrow \mathbb{C}$ .

(8 points)

5. Here is a statement of Morera's Theorem: *If  $f$  is continuous on a domain  $D$  and  $\oint_C f(z) dz = 0$  for every closed contour  $C$  in  $D$ , then  $f$  is analytic on  $D$ .* Your friend observes that  $-1/z$  is an antiderivative of  $1/z^2$  so  $\oint_C \frac{1}{z^2} dz = 0$  for every closed curve and then uses Morera's Theorem to conclude that  $1/z^2$  is entire. Is your friend correct? If not, what is wrong with your friend's reasoning? (6 points)

6. Do any two of the following four problems. Circle the problem numbers for the two problems you are submitting. (10 points each)

- (A) Prove the following: If  $f$  is analytic on and inside a simple closed contour  $C$  and  $z_0$  is not on  $C$ , then

$$\oint_C \frac{f''(z)}{z - z_0} dz = 2 \oint_C \frac{f(z)}{(z - z_0)^3} dz$$

- (B) Prove the following: If  $f$  is analytic on and inside a simple closed contour  $C$  and  $f(\tilde{z}) = 0$  for all  $\tilde{z}$  on  $C$ , then  $f(z) = 0$  for all  $z$  inside  $C$ .

- (C) Prove the following: If  $f$  is entire and  $\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 0$ , then  $f$  is constant.

- (D) Prove Morera's Theorem. (See Problem 5 for a statement of the theorem.)