Name ______________________________

MATH 352 Complex Analysis Spring 2003 Exam #4

Instructions: Do your own work. You may consult class notes, the course text, or other books. Give a reference if you use some source other than class notes or the course text. Turn in a complete and concise write up of your work. Show enough detail so that a peer could follow your work (both computations and reasoning). If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong or on alternate approaches you might try. The exam is due Wednesday, May 7 at 4:00 pm.

---

1. Consider a function $f(z) = \dfrac{p(z)}{q(z)}$ where p and q are analytic at z_0, $p(z_0) \neq 0$, and q has a zero of order 2 at z_0.
   (a) Show that f has a pole of order 2 at z_0 and find an expression for the residue of f at z_0 in terms of p, q, and derivatives of these functions.
   (b) Use your result to compute the residue of $f(z) = \dfrac{1}{\sin^2 z}$ at $z_0 = 0$.

2. Consider the contour integral $\displaystyle\oint_C \frac{z^4}{(z+1)(z-2)(z-2i)}\,dz$ where C is the circle of radius 3 centered at the origin.
   (a) Evaluate this using the usual residue theorem.
   (b) Evaluate this using Theorem 2 on page 185 of the text.

3. Problem #13 on page 216 of the text.

4. Find the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ by analyzing $\displaystyle\oint_{C_N} \frac{1}{z^2 \tan z}\,dz$ where C_N is the square contour with edges on the lines $x = \pm\left(N + \frac{1}{2}\right)\pi$ and $y = \pm\left(N + \frac{1}{2}\right)\pi$.

5. Show how to deduce the Cauchy-Goursat theorem and Cauchy's integral formulas starting from the residue theorem.