Some theory on systems of first-order linear homogeneous ODEs

We'll use the following notation:

- $C_n^1(a,b)$ is the vector space of column vector functions (of size n) with continuous first derivatives on the interval (a,b)
- $\vec{\theta}(t)$ is the zero column vector function (i.e., the column vector function for which each entry is identically zero for all t in (a,b))
- $[\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n]$ is the matrix with column vectors $\vec{A}_i$
- $W_S(t)$ is the Wronskian of the set $S = \{\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)\}$ defined as

$$W_S(t) = \det\big[\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)\big]$$

Theorem 1. Let $S = \{\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)\}$ be a set of functions in $C^n(a,b)$. If there is a t_0 in (a,b) such that $W_S(t_0)$ is nonzero, then S is linearly independent.

Proof. Start with the defining equation of linear independence

$$c_1\vec{f}_1(t) + c_2\vec{f}_2(t) + \cdots + c_n\vec{f}_n(t) = \vec{\theta}(t).$$

We must show that the only solution is the trivial solution. First we introduce some notation. Let $F(t) = [\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)]$. Let $\vec{c} = [c_1, c_2, \ldots, c_n]^T$. We can then write the defining equation as

$$F(t)\vec{c} = \vec{\theta}(t).$$

The Wronskian $W_S(t)$ is defined as the determinant of the coefficient matrix for this system. Since the Wronskian is nonzero for t_0 in (a,b), the system has a unique solution for that value t_0. This unique solution must be the trivial solution because the system of equations is homogeneous. Thus, the trivial solution is the only solution for all values of t. □

We now look at the set of solutions for a homogeneous system of n linear first-order differential equations.

Theorem 2. If $A(t)$ is an $n \times n$ matrix function that is continous on the interval (a,b), then the solution space $\mathcal{S} = \left\{\vec{y} \in C_n^1(a,b) \mid \dfrac{dy}{dt} = A\vec{y}\right\}$ is a subspace of $C_n^1(a,b)$ with dimension n.

Proof. It is straightforward to show that that $\mathcal{S}$ is a subspace of $C_n^1(a,b)$. One could do this directly or one could show that $\frac{d}{dt} - A(t)$ is a linear operator and recognize that S is the null space of this operator. To show that S has dimension n, we will find a basis with n elements.

To begin, we claim the existence of n solutions to the system by the existence-uniqueness theorem. In particular, pick some t_0 in (a,b) and let $\vec{h}_1(t)$, $\vec{h}_2(t)$, …, $\vec{h}_n(t)$ be the solutions which satisfy the initial conditions

$$\vec{h}_i(t_0) = \vec{e}_i$$

where $\vec{e}_i$ denotes the ith column of the $(n \times n)$ identity matrix I_n. Let $B = \{\vec{h}_1(t), \vec{h}_2(t), \ldots, \vec{h}_n(t)\}$. To prove that B is a basis for $\mathcal{S}$, we must show two things: one, that B is linearly independent; and two, that B spans $\mathcal{S}$.

To show linear independence, we note that $W_B(t_0) = \det(I_n) = 1 \neq 0$. By Theorem 1, the set B is linearly independent.

To prove that the set B spans $\mathcal{S}$, we must show that any other solution in $\mathcal{S}$ can be written as a linear combination of the elements in B. Let $\vec{y}(t)$ be any solution. For t_0, this solution has some value

$$\vec{y}(t_0) = \vec{c}$$

where $\vec{c} = [c_1, c_2, \ldots, c_n]^T$. Consider the solution given by the linear combination $c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + \cdots + c_n\vec{h}_n(t)$. Note that at t_0, this solution has the same value as the solution $\vec{y}(t)$. Hence, by the existence-uniqueness theorem, we have

$$\vec{y}(t) = c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + \cdots + c_n\vec{h}_n(t)$$

for all t in (a,b). This gives $\vec{y}(t)$ as a linear combination of the elements in B and thus completes the proof. □