Some matrix exponential problems

1. Use the definition of matrix exponential to directly compute e^{At} for $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ where λ is a constant.

2. Prove $(e^B)^{-1} = e^{-B}$ for any $(n \times n)$ matrix B. Note that $(e^B)^{-1}$ means the matrix inverse of e^B.

3. Compute $e^{At}\vec{w}$ where $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

   Hint: First write $\vec{w}$ as a linear combination of eigenvectors for A.

4. Consider the system $\dfrac{d\vec{y}}{dt} = A\vec{y}$ with $A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$.

   (a) Find the eigenstuff for A.

   (b) Construct a fundamental matrix solution for the system.

   (c) Compute e^{At}.

5. Prove the following: If $\Psi(t)$ is a fundamental matrix solution for $\dfrac{d\vec{y}}{dt} = A\vec{y}$, then

$$e^{A(t-t_0)} = \Psi(t)\Psi^{-1}(t_0).$$