Name ____________________

MATH 301 | Differential Equations | Fall 2005 | Final Exam

Instructions: You can work on the problems in any order. Please use just one side of each page and clearly number the problems. You do not need to write answers on the question sheet.

This exam is a tool to help me (and you) assess how well you are learning the course material. As such, you should report enough written detail for me to understand how you are thinking about each problem.

You can use integration aids such as a table of integrals.

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1. Show that $y(t) = -1$ is the only solution of the initial value problem (8 points)

$$\frac{dy}{dt} = 4t(1+y) \qquad y(0) = -1.$$

2. For each of the following, solve the given differential equation or initial-value problem. Express each result in terms of real-valued functions. (14 points each)

   (a) $\dfrac{dy}{dt} = 3t^2(y^2+1)$

   (b) $y'' - 10y' + 29y = 0, \quad y(0) = 2, \quad y'(0) = 4$

   (c) $y'' - 5y' + 6y = 3e^{4t} + \sin t$

   (d) $\dfrac{d}{dt}\vec{y} = A\vec{y}$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

3. Consider the system of equations (14 points)

$$\frac{dx}{dt} = x^2 + y^2 - 1$$
$$\frac{dy}{dt} = 2xy$$

   (a) Show that $(0, 1)$ is an equilibrium point for this system.

   (b) Find the relevant linearized system for this equilibrium point.

   (c) Analyze the linearized system in enough detail to draw a phase portrait. Sketch a phase portrait for the linearized system.

   (d) Determine if the linearized system is relevant to analyzing the original system.

4. Discuss similiarities between the structure of the general solution of a nonhomogeneous linear n^{th} order differential equation and the general solution of a nonhomogeneous system of n linear first order differential equations. (8 points)

There is more on the flip side.

5. Consider a species with a natural rate of change in population modeled as proportional to the population. If the population is harvested (think of fishing) at a constant rate h, a reasonable simple model is

$$\frac{dp}{dt} = ap - h \qquad p(0) = p_0.$$

Here, a, h, and p_0 are positive constants. (14 points)

(a) Sketch a slope field for $0 \leq t$ and $0 \leq y$ with enough detail to show all interesting features.

(b) On your slope field, sketch a solution curve for each of the following cases:
(i) $h < ap_0$ (ii) $h = ap_0$ (iii) $h > ap_0$.

(c) Explain what this model predicts about the population for each of the three cases
(i) $h < ap_0$ (ii) $h = ap_0$ (iii) $h > ap_0$.

(d) For the case $h > ap_0$, find the time T at which the population becomes 0.