Proof by induction

Proof by induction is a line of reasoning used to establish the truth of a list of statements $S(n)$ with n a natural number. For example, we might have

$$S(n): \qquad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

In this case, for each value of n, the statement $S(n)$ is an identity. As another example, consider

$S(n):$ If λ is an eigenvalue of the matrix A, then λ^n is an eigenvalue of the matrix A^n.

There are two steps in completing a proof by induction:

- Prove that $S(1)$ is true.

  This step is sometimes called *the base case*.

  In some cases, the list of statements $S(n)$ is to be proven for all natural numbers $n \geq K$ for a fixed K. In these cases, one proves that $S(K)$ is true.

- Prove the implication: If $S(k)$ is true, then $S(k+1)$ is true.

  The hypothesis "$S(k)$ is true" is sometimes referred to as *the inductive hypothesis.*

Note that in the second step, we need only prove that the implication holds for a generic value of k. The idea in proof by induction is that the first step establishes the truth of the first item in the list of statements and the second step establishes the validity of a link that says if one item in the list of statements is true, then the next item in the list is true. Together, these imply that all items in the list are true.

Problems
Use induction to prove each of the following.

1. For any natural number n, $2 + 4 + 6 + \cdots + (2n) = n^2 + n$.

2. If λ is an eigenvalue of the matrix A, then λ^n is an eigenvalue of the matrix A^n for any natural number n.

3. If A is an $(m \times m)$ matrix and S is a nonsingular $(m \times m)$ matrix, then $(S^{-1}AS)^n = S^{-1}A^nS$ for any natural number n.