Stokes' Theorem

Adding circulations

Consider two rectangular loops that share a common edge as shown below. Let C_1 be the blue loop and C_2 be the red loop with orientations as shown. Let C_3 be the loop that consists of going around the outside of the large rectangle formed by removing the common edge. We now claim that

$$\oint_{C1} \vec{F} \cdot d\vec{s} + \oint_{C_2} \vec{F} \cdot d\vec{s} = \oint_{C_3} \vec{F} \cdot d\vec{s} \tag{1}$$

for this situation. To see this is true, think of breaking each of these rectangular curves into four pieces. By the properties of line integrals, we can express $\oint_{C_1} \vec{F} \cdot d\vec{s}$ as a sum of four line integrals, one over each side of the first rectangle. Likewise, we can express $\oint_{C_2} \vec{F} \cdot d\vec{s}$ as a sum of four line integrals, one over each side of the second rectangle. The sum $\oint_{C1} \vec{F} \cdot d\vec{s} + \oint_{C_2} \vec{F} \cdot d\vec{s}$ will have a total of eight terms. Two of these will involve the common edge. Since this is traversed once in each direction, these terms cancel. The remaining six terms can be put together to give $\oint_{C_3} \vec{F} \cdot d\vec{s}$.

This result is easily generalized to any polygonal loops that share a common edge.

Stokes' Theorem

Let $\vec{F}$ be a vector field in space. Let S be a two-sided surface in the domain of $\vec{F}$ with area vectors $d\vec{A}$ all on the same side. Let C be the curve in space that forms the edge of S. Choose the orientation of C that is compatible with the choice of side for $d\vec{A}$. If $\vec{F}$, S, and C are "nice" (in a specific technical sense), then

$$\iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{s}. \tag{2}$$

We give an outline of the proof in the following steps.

1. Start by considering the definition of surface integral. Break the surface S into pieces ΔS_{ij} having area vectors $\Delta \vec{A}_{ij} = \Delta A_{ij} \hat{n}_{ij}$. (See the figure that follows.) For each piece, pick a point $\mathcal{P}_{ij}$ at which to evaluate the vector field. Also, let

ΔC_{ij} be the curve that forms the edge of the ij-th piece. Use the definition of surface integral to write

$$\oiint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \lim_{\Delta A \to 0} \sum_{j=1}^{n} \sum_{i=i}^{m} \vec{\nabla} \times \vec{F}(\mathcal{P}_{ij}) \cdot \Delta\vec{A}_{ij}. \tag{3}$$

2. Next, recall that $(\vec{\nabla} \times \vec{F}) \cdot \hat{n}$ is defined as the circulation density for a planar region with $d\vec{A} = dA\hat{n}$. Thus, for a small loop, we have

$$\oint\limits_{\Delta C} \vec{F} \cdot d\vec{s} \approx (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\Delta A = (\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{A} \tag{4}$$

That is, for a small loop, the circulation is approximately $(\vec{\nabla} \times \vec{F}) \cdot \Delta\vec{A}$. Substitute into Equation (3) using Equation (4) to get

$$\oiint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \lim_{\Delta A \to 0} \sum_{j=1}^{n} \sum_{i=i}^{m} \oint\limits_{\Delta C_{ij}} \vec{F} \cdot d\vec{s}.$$

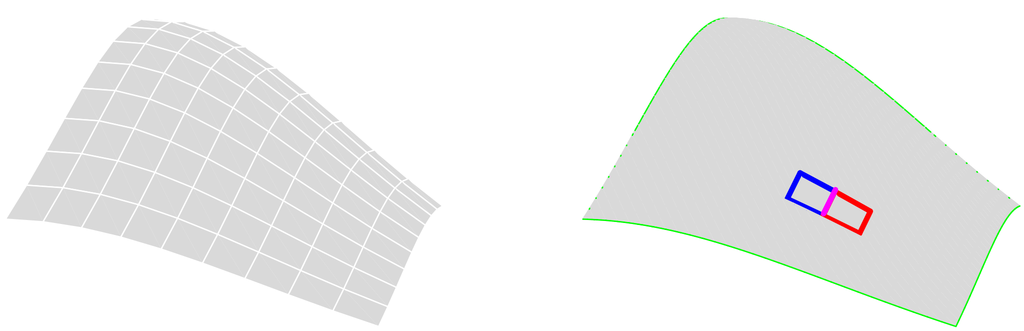
3. The sum of line integrals on the left is over curves ΔC_{ij} with lots of common edges. Using the result on adding circulations, we see that all of the contributions from the interior edges cancel since each is traversed twice, once in each direction. The net result is from the contributions on the exterior edges which is equivalent to the line integral over the original curve C. That is,

$$\sum_{j=1}^{n} \sum_{i=i}^{m} \oint\limits_{\Delta C_{ij}} \vec{F} \cdot d\vec{s} = \oint\limits_C \vec{F} \cdot d\vec{s}.$$

Substituting this gives

$$\oiint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint\limits_C \vec{F} \cdot d\vec{s}.$$

and we are done.



The technical details that are missing here justify the approximation we use in Step 2.