

Notes on line integrals

Version 2 (April 22, 2004)

Definition of line integral

We are given a vector field \vec{F} and a curve C in the domain of \vec{F} . The general idea of line integral is

line integral of \vec{F} over curve C =
 the limit of a sum of terms each having the form
 (component of \vec{F} tangent to C)(length of piece of C).

Here's how we make the idea precise. Break the curve C into n pieces with endpoints P_1, P_2, \dots, P_{n+1} . (See Figure 1 at the end.) We can refer to these as P_i with the index i ranging from 1 to $n + 1$. Define $\Delta\vec{R}_i$ to be the displacement between point P_i and point P_{i+1} . (See Figure 2.) That is, $\Delta\vec{R}_i = \vec{P}_i\vec{P}_{i+1}$. At each of the points, compute the vector field output $\vec{F}(P_i)$. Recall that the dot product $\vec{F}(P_i) \cdot \Delta\vec{R}_i$ can be written

$$\vec{F}(P_i) \cdot \Delta\vec{R}_i = \|\vec{F}(P_i)\| \|\Delta\vec{R}_i\| \cos \theta = \left(\|\vec{F}(P_i)\| \cos \theta \right) \|\Delta\vec{R}_i\|.$$

The last expression shows that this dot product gives the component of \vec{F} tangent to C times the length of a piece of C . This is what we want to add up. We define the line integral of \vec{F} for the curve C as the limit of such a sum:

$$\int_C \vec{F} \cdot d\vec{R} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{F}(P_i) \cdot \Delta\vec{R}_i$$

You can think of $d\vec{R}$ as an "infinitesimal" version of $\Delta\vec{R}_i$. The direction of $d\vec{R}$ is tangent to the curve at each point. (See Figure 3.)

Notation

The text often uses an alternate notation for the line integral. Here's the connection: Write the vector field \vec{F} in terms of components as $\vec{F} = u\hat{i} + v\hat{j} + w\hat{k}$ and write the vector $d\vec{R}$ in terms of components as $d\vec{R} = dx\hat{i} + dy\hat{j} + dz\hat{k}$. Here, think of dx as a small displacement parallel to the x -axis, dy as a small displacement parallel to the y -axis, and dz as a small displacement parallel to the z -axis. With these component expressions, we can write out the dot product as

$$\vec{F} \cdot d\vec{R} = (u\hat{i} + v\hat{j} + w\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = u dx + v dy + w dz.$$

Using this, the notation for line integral can be written

$$\int_C \vec{F} \cdot d\vec{R} = \int_C u dx + v dy + w dz.$$

The text favors the expression on the right side and I generally use the expression on the left side.

Most of the problems are given using the notation on right side. For example, Problem 7 of Section 13.2 gives the line integral

$$\int_C (-y dx + x dy).$$

From this, you can read off that the vector field is $\vec{F} = -y \hat{i} + x \hat{j}$.

Computing line integrals

In computing line integrals, the general plan is to express everything in terms of a single variable. This is a reasonable thing to do because a curve is a one-dimensional object. The essential things are to determine the form of $d\vec{R}$ for the curve C and the outputs $\vec{F}(P)$ along the curve C , all in terms of one variable. The displacement $d\vec{R}$ is defined to have components

$$d\vec{R} = dx \hat{i} + dy \hat{j}$$

How to proceed depends on how we describe the curve. In general, we have two choices: a relation between the coordinates or a parametric description. The two solutions to the following example show how to work with each of these.

Example: Compute the line integral of $\vec{F}(x, y) = 3\hat{i} + 2\hat{j}$ for the curve C that is the upper half of the circle of radius 1 traversed from left to right.

Note: To get started, you should draw a picture showing the curve and a few of the vector field outputs along the curve.

Solution 1: The equation of the circle is $x^2 + y^2 = 1$. From this, we compute

$$2x dx + 2y dy = 0.$$

Solving for dy and substituting from $x^2 + y^2 = 1$ gives

$$dy = -\frac{x}{y} dx = -\frac{x}{\sqrt{1-x^2}} dx.$$

This is the relation between dx and dy for a displacement $d\vec{R}$ along the circle. Substituting this gives

$$d\vec{R} = dx \hat{i} + dy \hat{j} = dx \hat{i} - \frac{x}{\sqrt{1-x^2}} dx \hat{j} = \left(\hat{i} - \frac{x}{\sqrt{1-x^2}} \hat{j} \right) dx$$

The vector field here is constant so all outputs along the curve C are $\vec{F}(P) = 3\hat{i} + 2\hat{j}$. We thus have

$$\vec{F} \cdot d\vec{R} = \left(3\hat{i} + 2\hat{j} \right) \cdot \left(\hat{i} - \frac{x}{\sqrt{1-x^2}} \hat{j} \right) dx = \left(3 - \frac{2x}{\sqrt{1-x^2}} \right) dx$$

This is the integrand. For the curve C , the variable x ranges from -1 to 1 , so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_{-1}^1 \left(3 - \frac{2x}{\sqrt{1-x^2}} \right) dx = \text{some work to be done here} = 6.$$

Solution 2: We parametrize the curve by

$$x = -\cos t \quad \text{and} \quad y = \sin t \quad \text{for} \quad 0 \leq t \leq \pi.$$

You should confirm that this traces out the curve C in the correct direction (from left to right). From these, we compute

$$dx = \sin t \, dt \quad \text{and} \quad dy = \cos t \, dt.$$

Substituting into $d\vec{R}$ gives

$$d\vec{R} = dx \hat{i} + dy \hat{j} = \sin t \, dt \hat{i} + \cos t \, dt \hat{j} = (\sin t \hat{i} + \cos t \hat{j}) \, dt.$$

The vector field here is constant so all outputs along the curve C are $\vec{F}(P) = 3\hat{i} + 2\hat{j}$. We thus have

$$\vec{F} \cdot d\vec{R} = (3\hat{i} + 2\hat{j}) \cdot (\sin t \hat{i} + \cos t \hat{j}) \, dt = (3 \sin t + 2 \cos t) \, dt.$$

This is the integrand. For the curve C , the variable t ranges from 0 to π , so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_0^\pi (3 \sin t + 2 \cos t) \, dt = \text{some work to be done here} = 6.$$

Comments: In comparing the two solutions, you might think that the algebra is more complicated in Solution 1. This is probably so. The advantage of Solution 1 is that we all know the equation of a circle is $x^2 + y^2 = 1$. For Solution 2, to get started, we need to parametrize the curve. This is not too bad for a circle. The choice of which style to use depends on personal preference and the easiest way to describe a given curve.

If you have corrections or suggestions for improvement, please contact Martin Jackson, Department of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA 98416, martinj@ups.edu.

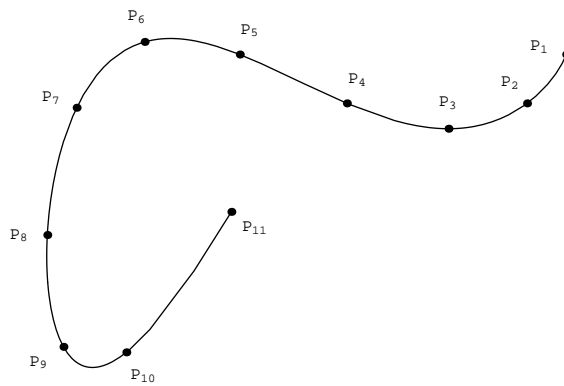


Figure 1. The curve C broken into pieces with endpoint P_i .

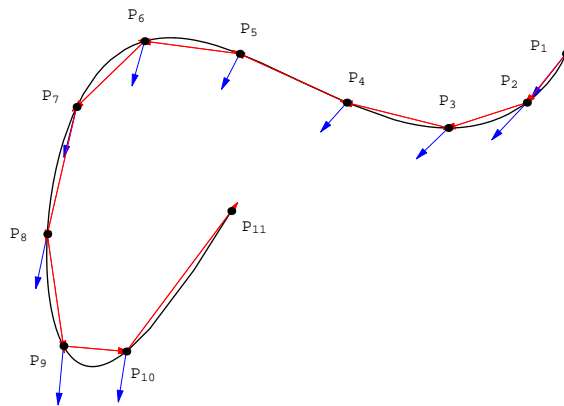


Figure 2. The curve C with the vectors $\Delta \vec{R}_i$ (in red) and $\vec{F}(P_i)$ (in blue).

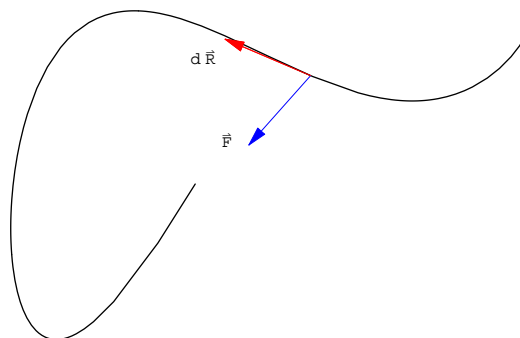


Figure 3. The curve C with an example of $d\vec{R}$ and \vec{F} at a point.