# Divergence and curl in other coordinate systems

To get expressions for divergence and curl in cylindrical and spherical coordinates, it is easiest to start by deriving more general expressions and then make substitutions.

Start by considering generic coordinates $(x_1, x_2, x_3)$ for space. Now consider a box defined by small changes of extent $dx_1$, $dx_2$, and $dx_3$. This box will have edge lengths corresponding to each of the changes $dx_i$. Let $h_1 dx_1$ be the length of the edge corresponding to a change from $x_1$ to $x_1 + dx_1$. Similarly, let $h_2 dx_2$ and $h_3 dx_3$ be the lengths of the edges corresponding to changes in $x_2$ and $x_3$, respectively.

**Example** For cylindrical coordinates, we have $x_1 = r$, $x_2 = \theta$ and $x_3 = z$. From our previous experience, we know that the corresponding edge lengths are

$$h_1 dx_1 = dr, \qquad h_2 dx_2 = r d\theta, \qquad \text{and} \qquad h_3 dx_3 = dz.$$

Thus, $h_1 = 1$, $h_2 = r$, and $h_3 = 1$ for cylindrical coordinates.

**Exercise** Write down the edge length expressions for spherical coordinates.

Note, in general, that each $h_i$ is a function of the coordinates so we should properly write $h_i(x_1, x_2, x_3)$. You should do so when it is important to be explicit about inputs.

Here's the strategy:

1. Work out general expressions for divergence and curl in terms of the coordinates $(x_1, x_2, x_3)$ and the "length" functions $h_1$, $h_2$, and $h_3$ for a vector field with components given by
$$\vec{F} = F_1\,\hat{x}_1 + F_2\,\hat{x}_2 + F_3\,\hat{x}_3$$
where each of the components $F_i$ is a function of the coordinates $(x_1, x_2, x_3)$.

2. For each coordinate system, substitute the relevant edge length expressions and simplify.

The expressions you work out for Step 1 are not as bad as you might imagine right now. Working with generic coordinates and edge length functions will produce expressions that have lots of symmetry in the indices 1, 2, and 3.

In simplifying the expressions you get in Step 2, take advantage of the fact that other coordinates are constant in partial derivatives with respect to one coordinate. For example, in spherical coordinates we have

$$\frac{1}{r^2 \sin\phi}\frac{\partial}{\partial\theta}\big(r\sin\phi F_\theta\big) = \frac{1}{r^2\sin\phi} r\sin\phi \frac{\partial}{\partial\theta}\big(F_\theta\big) = \frac{1}{r}\frac{\partial}{\partial\theta}\big(F_\theta\big) = \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}$$