Curl and circulation density

Circulation

Given a vector field $\vec{F}$ and an oriented closed loop C in space, we can think of the line integral $\oint_C \vec{F} \cdot d\vec{s}$ as a *circulation*. In this interpretation, we think of $\vec{F}$ as the velocity field of a fluid flow and think about the curve C as a rigid wire in this flow. Now think about pushing a bead that is strung on the wire. The fluid flow may help or hinder as we push the bead around the entire loop. The line integral sums up the tangential component of the velocity times a displacement along the curve. At a point where the tangential component of the velocity is in the same direction as the displacement, the contribution to the line integral is positive. This corresponds to the fluid flow helping us in pushing the bead. At a point where the tangential component of the velocity is in the opposite direction to the displacement, the contribution to the line integral is negative. This corresponds to the fluid flow hindering us in pushing the bead. The line integral gives the net help along the entire curve. Thus, circulation is a measure of how much the flow helps if we push a bead around the wire in the direction specified by the orientation. If the circulation is positive, the fluid flow is a net help. If the circulation is negative, the fluid flow is a net hinderance.

Circulation density and curl

Start with a vector field $\vec{F}$ and focus on a point $\mathcal{P}$ in the domain of the vector field. Imagine a small, flat region that contains $\mathcal{P}$. (You can think of a rectangle or disk if it helps to be specific about the shape.) We will use ΔD to denote this planar region. Here Δ doesn't mean "a small change in" but serves to remind us that the region is small. Let $\Delta\vec{A}$ be the area vector for ΔD. Let ΔC be the closed curve that is the edge of this region. Orient the curve ΔC so that the thumb of your right hand points in the direction of $\Delta\vec{A}$ when your fingers curl around in the direction of ΔC.

Both the circulation $\oint_{\Delta C} \vec{F} \cdot d\vec{s}$ and the area ΔA will go to zero as we shrink the curve ΔC down to the point $\mathcal{P}$. However, the limit of the ratio

$$\frac{\displaystyle\oint_{\Delta C} \vec{F} \cdot d\vec{s}}{\Delta A}$$

might exist. This limit is the *circulation density*. That is, the limit of the ratio is the circulation per unit area. We define the *curl of the vector field* $\vec{F}$ in terms of this circulation density. The curl of $\vec{F}$ is itself a vector field. The following definition gives the $\hat{n}$-component of the curl of $\vec{F}$ as a circulation density.

> *The $\hat{n}$-component of the curl of $\vec{F}$ at $\mathcal{P}$ is defined as the circulation density at a point $\mathcal{P}$ for a loop with $d\vec{A} = dA\,\hat{n}$. That is,*
>
> $$(\operatorname{curl}\vec{F}) \cdot \hat{n} = \lim_{\text{“}\Delta C \to \mathcal{P}\text{''}} \frac{\displaystyle\oint_{\Delta C} \vec{F} \cdot d\vec{s}}{\Delta A} \tag{1}$$

Figure 1 shows a curve used in calculating circulation density and the curl of the vector field. *(Need more detail here relating circulation density to a "paddlewheel" interpretation.)*

An expression for curl in cartesian coordinates

From the definition of curl in terms of circulation density, we learn what curl tells us about the vector field. However, computing the curl from this definition is difficult. We'll next look at getting an expression for the curl in terms of partial derivatives with respect to cartesian coordinates.

Let the vector field $\vec{F}$ be given in cartesian coordinates by

$$\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}.$$

The curl of $\vec{F}$ is itself a vector field so we are looking for an expression of the form

$$\operatorname{curl}\vec{F} = (\hat{\imath}\text{-component})\hat{\imath} + (\hat{\jmath}\text{-component})\hat{\jmath} + (\hat{k}\text{-component})\hat{k}.$$

We'll look at the $\hat{k}$-component in detail and leave the others as exercises. Let ΔD be a small rectangle parallel to the xy-plane with $\Delta\vec{A} = \Delta A\hat{k}$. Then ΔC is oriented counter-clockwise as viewed from above. (See Figure 2.) Let $\mathcal{P}$ be one corner of this rectangle. Let Δx and Δy be the side lengths parallel to the x-axis and y-axis respectively. The area of the rectangle is $\Delta A = \Delta x\,\Delta y$. In Figure 2, the area vector $\Delta\vec{A}$ is shown in red.

Now let's think about the circulation for this rectangle as given by the line integral $\oint_{\Delta C}\vec{F}\cdot d\vec{s}$. Since we will be eventually taking a limit as the rectangle shrinks to the point $\mathcal{P}$, we can approximate this line integral with just four terms, one for each side of the rectangle. Let $\Delta\vec{s}_1$, $\Delta\vec{s}_2$, $\Delta\vec{s}_3$, and $\Delta\vec{s}_4$ be the displacement vectors along the four sides as shown in Figure 3. For each side, we can choose where to evaluate the vector field $\vec{F}$. We will use the corners $\mathcal{P}$, $\mathcal{P}_1$, and $\mathcal{P}_2$ shown in Figure 2. Specifically, we use

$$\oint_{\Delta C}\vec{F}\cdot d\vec{s} \approx \vec{F}(\mathcal{P})\cdot\Delta\vec{s}_1 + \vec{F}(\mathcal{P}_1)\cdot\Delta\vec{s}_2 + \vec{F}(\mathcal{P}_2)\cdot\Delta\vec{s}_3 + \vec{F}(\mathcal{P})\cdot\Delta\vec{s}_4. \tag{2}$$

Now let's introduce coordinates and components for the vectors in Equation (2). The vector field $\vec{F}$ has components $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$. We choose coordinates with $\mathcal{P} = (x,y,z)$. Since the side lengths are Δx and Δy, this gives us coordinates $\mathcal{P}_1 = (x+\Delta x, y, z)$ and $\mathcal{P}_2 = (x, y+\Delta y, z)$. From the geometry, we can see that $\Delta\vec{s}_1 = \Delta x\,\hat{\imath}$. Thus

$$\vec{F}(\mathcal{P})\cdot\Delta\vec{s}_1 = \Big(P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}\Big)\cdot\big(\Delta x\,\hat{\imath}\big) = P(x,y,z)\Delta x.$$

The other three terms are similar:

$$\begin{aligned}
\Delta\vec{s}_2 &= \Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(\mathcal{P}_1)\cdot\Delta\vec{s}_2 = Q(x, y+\Delta x, z)\Delta y \\
\Delta\vec{s}_3 &= -\Delta x\,\hat{\imath} \quad \text{so} \quad \vec{F}(\mathcal{P}_2)\cdot\Delta\vec{s}_3 = -P(x, y+\Delta y, z)\Delta x \\
\Delta\vec{s}_4 &= -\Delta y\,\hat{\jmath} \quad \text{so} \quad \vec{F}(\mathcal{P})\cdot\Delta\vec{s}_4 = -Q(x,y,z)\Delta y
\end{aligned}$$

Stop here and make sure you understand how to get these expressions including the negative signs in the last two of these.

Now substitute into Equation (2) to get

$$\begin{aligned}
\oint_{\Delta C} \vec{F} \cdot d\vec{s} &\approx P(x,y,z)\Delta x + Q(x,y+\Delta x,z)\Delta y - P(x,y+\Delta y,z)\Delta x - Q(x,y,z)\Delta y \\
&= \big[P(x,y,z) - P(x,y+\Delta y,z)\big]\Delta x + \big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y \\
&= \big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y - \big[P(x,y+\Delta y,z) - P(x,y,z)\big]\Delta x
\end{aligned}$$

Thus, the ratio of circulation to area enclosed is

$$\begin{aligned}
\frac{\oint_{\Delta C} \vec{F} \cdot d\vec{s}}{\Delta A} &= \frac{\big[Q(x,y+\Delta x,z) - Q(x,y,z)\big]\Delta y - \big[P(x,y+\Delta y,z) - P(x,y,z)\big]\Delta x}{\Delta x\, \Delta y} \\
&= \frac{Q(x,y+\Delta x,z) - Q(x,y,z)}{\Delta x} - \frac{P(x,y+\Delta y,z) - P(x,y,z)}{\Delta y}. \qquad (3)
\end{aligned}$$

Now consider the limit as ΔC shrinks to the point $\mathcal{P}$. We achieve this by taking $\Delta x \to 0$ and $\Delta y \to 0$. Note each of the two terms on the right side of Equation (3) will give a partial derivative in this limit. We thus have

$$\hat{k}\text{-component of } \operatorname{curl} \vec{F} \text{ at } \mathcal{P} \ = \lim_{\Delta x, \Delta y \to 0} \frac{\oint_{\Delta C} \vec{F} \cdot d\vec{s}}{\Delta A} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \qquad (4)$$

for an "infinitesimal loop" parallel to the xy-plane. Another way to specify this orientation is to say the area vector for this "infinitesimal loop" is $d\vec{A} = dA\,\hat{k}$.

Exercise: Compute the $\hat{\imath}$- and $\hat{\jmath}$-components of the curl of $\vec{F}$. The answers are

$$\hat{\imath}\text{-component of } \operatorname{curl} \vec{F} \text{ at } \mathcal{P} \ = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \qquad (5)$$

$$\hat{\jmath}\text{-component of } \operatorname{curl} \vec{F} \text{ at } \mathcal{P} \ = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}. \qquad (6)$$

Putting these together, we have the following result.

In cartesian coordinates, the curl of

$$\vec{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

is given by

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}. \qquad (7)$$

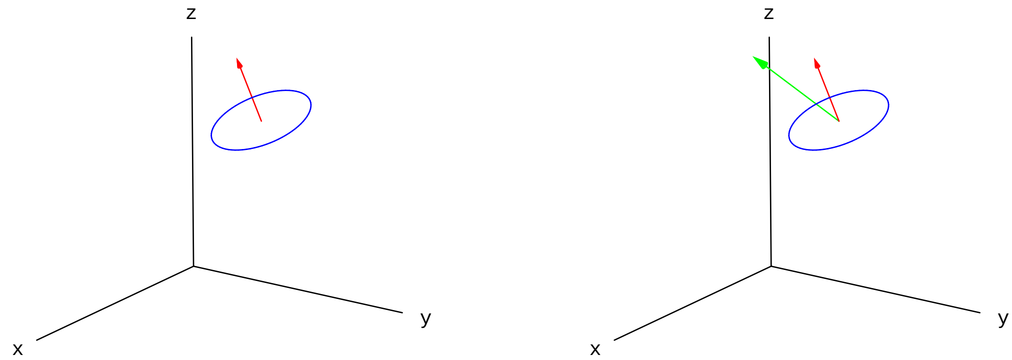

Figure 1. Left: A small flat region ΔD with area vector $\Delta\vec{A} = \Delta A\,\hat{n}$ in red.
Right: Same as left with the addition of the curl of $\vec{F}$ at $\mathcal{P}$ (in green).

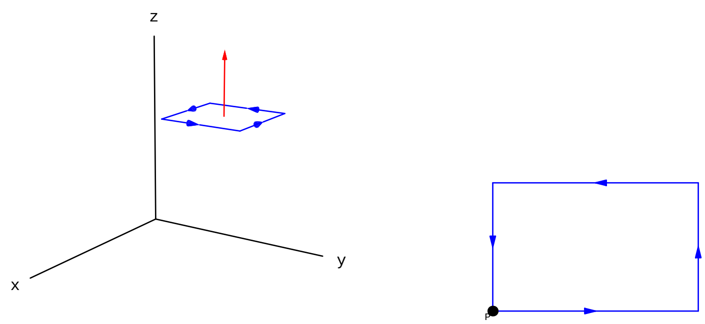

Figure 2. The small rectangle ΔC shown in perspective (left) and from above (right).

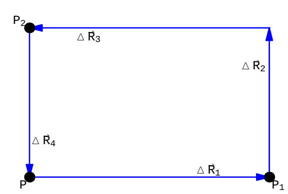

Figure 3. The points and edges of the small rectangle ΔC.

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