Notes on surface integrals

Version 1.1 (April 24, 2004)

Definition of surface integral

We are given a vector field $\vec{F}$ in space and a surface S in the domain of $\vec{F}$. The general idea of surface integral is

surface integral of $\vec{F}$ over surface S=
the limit of a sum of terms each having the form
(component of $\vec{F}$ normal to a piece S)(area of that piece of S).

Here's how we make the idea more precise. Break the surface S into pieces and lablel these pieces ΔS_{ij}. We use two indices because the surface is a two-dimensional thing. Think of the index i as running from 1 to m and the index j as running from 1 to n for a total of mn pieces. (See Figure 1 at the end.) Define $\Delta\vec{A}_{ij}$ as the vector with direction normal to the piece ΔS_{ij} and magnitude equal to the area of the piece ΔS_{ij}. In each piece, pick a point P_{ij}. At each of the points, compute the vector field output $\vec{F}(P_{ij})$. Recall that the dot product $\vec{F}(P_{ij}) \cdot \Delta\vec{A}_{ij}$ can be written

$$\vec{F}(P_{ij}) \cdot \Delta\vec{A}_{ij} = \|\vec{F}(P_{ij})\|\|\Delta\vec{A}_{ij}\| \cos\theta = \Big(\|\vec{F}(P_{ij})\| \cos\theta\Big)\|\Delta\vec{A}_{ij}\|.$$

The last expression shows that this dot product gives the component of $\vec{F}$ normal to ΔS_{ij} times the area of ΔS_{ij}. This is what we want to add up. We define the surface integral of $\vec{F}$ for the surface S as the limit of such a sum:

$$\iint_S \vec{F} \cdot d\vec{A} = \lim_{m,n\to\infty} \sum_{i=j}^{n} \sum_{i=1}^{n} \vec{F}(P_{ij}) \cdot \Delta\vec{A}_{ij}$$

You can think of $d\vec{A}$ as an "infinitesimal" version of $\Delta\vec{A}_{ij}$. The direction of $d\vec{A}$ is normal to the surface at each point.

In order to compute a surface integral, it is useful to think of $d\vec{A}$ in the following way. Consider the "grid" on the surface S (as shown in Figure 1 at the end) as consisting of two families of curves, with each family consisting of those curves that are *locally* parallel, but not *globally* parallel in general. Here, locally parallel means that the family of curves is parallel if we zoom in on any point. An important thing here is that within each family, no two curves intersect. The curves in the first family do not have to be perpendicular to the curves in the second family. At a point P on the surface, take C_1 to be the curve from one family that goes through P and C_2 to be the curve from the other family that goes through P. Let $d\vec{R}_1$ be the infinitesimal displacement vector tangent to C_1 at P and let $d\vec{R}_2$ be the infinitesimal displacement vector tangent to C_2 at P. Consider the cross product $d\vec{R}_1 \times d\vec{R}_2$. Recall the geometric definition of cross product: (1) the direction of the cross product is perpendicular to

both vectors in the product (as given by the right hand rule); and (2) the magnitude of the cross product is the area of the parallelogram formed by the two vectors in the product. The cross product $d\vec{R}_1 \times d\vec{R}_2$ is thus normal to the surface at P (since $d\vec{R}_1$ and $d\vec{R}_2$ are both tangent to the surface) and has magnitude equal to the area of the surface piece with edges $d\vec{R}_1$ and $d\vec{R}_2$. Thus

$$d\vec{A} = d\vec{R}_1 \times d\vec{R}_2.$$

Computing surface integrals

In computing line integrals, the general plan is to express everything in terms of two variables. This is a reasonable thing to do because a surface is a two-dimensional object. The essential things are to determine the form of $d\vec{A}$ for the surface S and the outputs $\vec{F}(P)$ along the surface S, all in terms of two variables. How to proceed depends on how we describe the curve. In general, we have two choices: a relation among the coordinates or a parametric description. The solution to the following example illustrates how to work with the first of these.

Example: Compute the line integral of $\vec{F}(x, y, z) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ for the surface S that is the piece of the plane $12x - 6y + 3z = 24$ with $x \geq 0$, $y \leq 0$, and $z \geq 0$.

Note: To get started, you should draw a picture showing the surface and a few of the vector field outputs along the surface.

Solution : From the equation of the plane, we compute

$$12\,dx - 6\,dy + 3\,dz = 0.$$

This relates small displacements dx, dy, and dz along the plane. See Figure 2.

To generate one family of curves on the surface, we can use $x =$ constant. This gives us $dx = 0$. Using this in the previous relation and solving for dz gives $dz = 2dy$. We can now use these to get

$$d\vec{R}_1 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = 0\,\hat{\imath} + dy\,\hat{\jmath} + 2dy\,\hat{k} = \left(\hat{\jmath} + 2\,\hat{k}\right) dy.$$

To generate the other family of curves on the surface, we can use $y =$ constant. This gives us $dy = 0$. Using this in the above relation and solving for dz gives $dz = -4dx$. We can now use these to get

$$d\vec{R}_2 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = dx\,\hat{\imath} + 0\,\hat{\jmath} - 4dx\,\hat{k} = \left(\hat{\imath} - 4\,\hat{k}\right) dx.$$

With $d\vec{R}_1$ and $d\vec{R}_2$ in hand, we can compute $d\vec{A}$ as

$$d\vec{A} = d\vec{R}_1 \times d\vec{R}_2 = \left(\hat{\jmath} + 2\,\hat{k}\right) \times \left(\hat{\imath} - 4\,\hat{k}\right) dxdy = \left(-6\,\hat{\imath} + 3\,\hat{\jmath} - \hat{k}\right) dxdy.$$

You should think about the direction these vectors point. We have made choices that result in $d\vec{A}$ pointing in a certain direction. A different set of choices could result in $d\vec{A}$ pointing in the opposite direction.

We now want to express the vector field outputs along the surface S in terms of the same two variables (x and y in this case) that we have used for $d\vec{A}$. We will use the equation of the plane to solve for z giving

$$z = 8 - 4x + 2y.$$

Thus, on the surface, the vector field has outputs

$$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + (8 - 4x + 2y)\,\hat{k}.$$

We now compute

$$\begin{aligned}
\vec{F} \cdot d\vec{A} &= \big(x\,\hat{\imath} + y\,\hat{\jmath} + (8 - 4x + 2y)\,\hat{k}\big) \cdot \big(-6\,\hat{\imath} + 3\,\hat{\jmath} - \hat{k}\big)\,dxdy \\
&= \big(-6x + 3y - (8 - 4x + 2y)\big)\,dxdy \\
&= \big(-2x + y - 8\big)\,dxdy
\end{aligned}$$

The last things we need in order to carry out the integration are the relevant bounds on the variables x and y. The projection of the surface into the xy-plane is the triangular region shown in Figure 3. We can use

$$0 \le x \le 2 \qquad \text{and} \qquad 2x - 4 \le y \le 0$$

to describe this region.

Putting all of this together, we have

$$\iint_S \vec{F} \cdot d\vec{A} = \int_0^2 \int_{4-2x}^0 \big(-2x + y - 8\big)\,dxdy = \text{ some work to do here } = -32.$$

The sign here is a result of the choices we made in computing $d\vec{A}$. A different set of choices could result in the value $+32$. You should think about the two possible directions for $d\vec{A}$.

If you have corrections or suggestions for improvements to these notes, please contact Martin Jackson, Department of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA 98416, martinj@ups.edu.

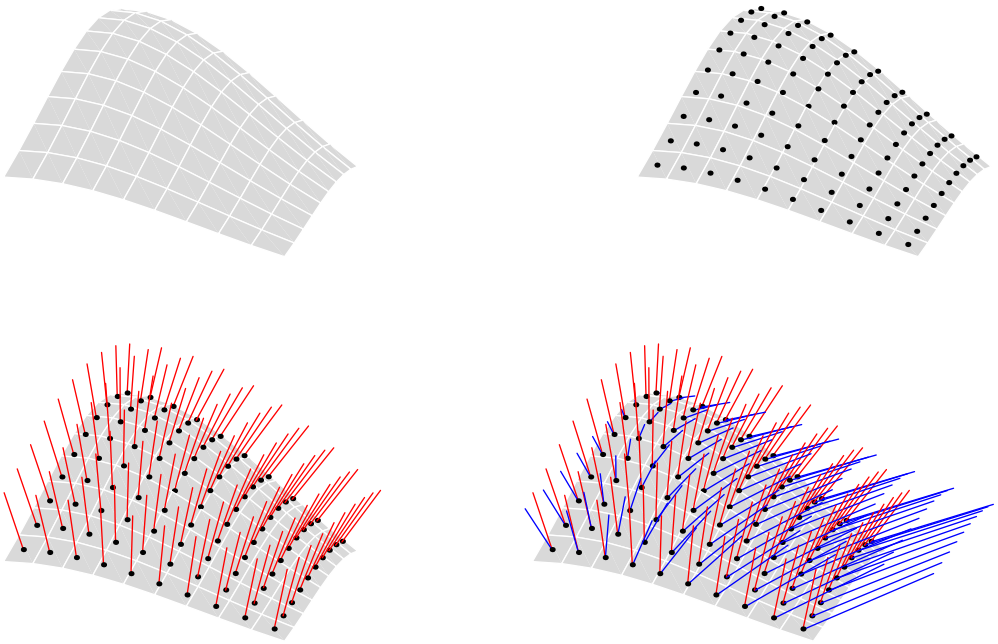

Figure 1. The elements used in the definition of surface integral. Top left: The surface S broken into pieces ΔS_{ij}. Top right: The points P_{ij}. Bottom left: The area vectors ΔA_{ik}. Bottom right: The vector field outputs $\vec{F}(P_{ij})$. Note that vectors are displayed without arrow heads to reduce clutter. The base of each vector is on the surface.

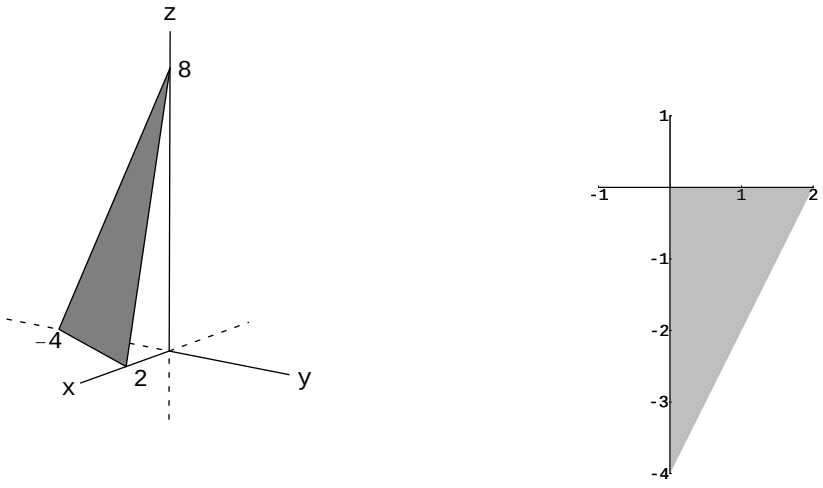

Figure 2. The piece of the plane that is the surface for the example (right) and the projection of this plane into the xy -plane (left).